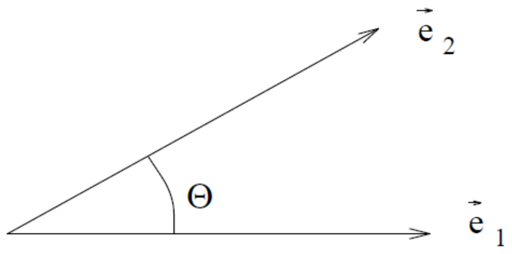
Lecture 7 - 050218



Example **2.2.5 (Non-diagonal basis-vectors)**

$$\vec{e}_1 \cdot \vec{e}_1 = 1, \qquad \vec{e}_2 \cdot \vec{e}_2 = 1, \qquad \vec{e}_1 \cdot \vec{e}_2 = \cos\theta = \vec{e}_2 \cdot \vec{e}_1$$
$$g_{\mu\nu} = \begin{pmatrix} 1 & \cos\theta \\ \cos\theta & 1 \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \tag{2.70}$$

Definition 2.2.4 (Contravariant components)
The **contravariant components** $g^{\mu\alpha}$ of the metric tensor are defined as:

$$g^{\mu\alpha} g_{\alpha\nu} \equiv \delta^{\mu}{}_{\nu} \quad g^{\mu\nu} = \vec{w}^{\mu} \cdot \vec{w}^{\nu}, \tag{2.71}$$

where $\vec{w}^{\mu}$ is defined by

$$\vec{w}^{\mu} \cdot \vec{w}_{\nu} \equiv \delta^{\mu}{}_{\nu}. \tag{2.72}$$

$g^{\mu\nu}$ is the inverse matrix of $g_{\mu\nu}$.

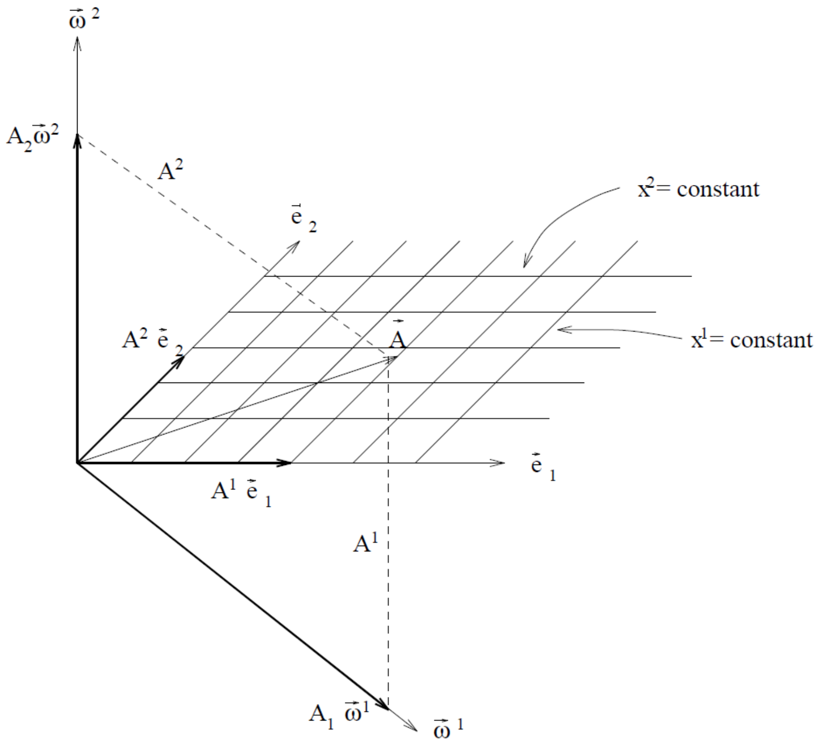


Figure 2.9: The covariant- and contravariant components of a vector

It is possible to define a **mapping** between tensors of different type (eg. covariant on contravariant) using the metric tensor.

We can for instance map a vector on a 1-form:

$$v_\mu = g(\vec{v}, \vec{e}_\mu) = g(v^\alpha \vec{e}_\alpha, \vec{e}_\mu) = v^\alpha g(\vec{e}_\alpha, \vec{e}_\mu) = v^\alpha g_{\alpha\mu} \tag{2.73}$$

This is known as lowering of an index. Raising of an index becomes :

$$v^\mu = g^{\mu\alpha} v_\alpha \tag{2.74}$$

The mixed components of the metric tensor becomes:

$$g^\mu{}_\nu = g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu_\nu \tag{2.75}$$

We now define distance along a curve. Let the curve be parameterized by λ (proper-time τ for time-like curves). Let $\vec{v}$ be the tangent vector-field of the curve.

The squared distance ds^2 between the points along the curve is defined as:

$$ds^2 \equiv g(\vec{v}, \vec{v}) d\lambda^2 \tag{2.76}$$

gives

$$ds^2 = g_{\mu\nu} v^\mu v^\nu d\lambda^2. \tag{2.77}$$

The tangent vector has components $v^\mu = \frac{dx^\mu}{d\lambda}$, which gives:

$$\boxed{ds^2 = g_{\mu\nu} dx^\mu dx^\nu} \tag{2.78}$$

The expression ds^2 is known as the **line-element**.

Example 2.2.6 (Cartesian coordinates in a plane)

$$\begin{aligned} g_{xx} = g_{yy} = 1, \qquad g^x{}_y = g^y{}_x = 0 \\ ds^2 = dx^2 + dy^2 \end{aligned} \tag{2.79}$$

Example 2.2.7 (Plane polar coordinates)

$$\begin{aligned} g_{rr} = 1, \quad g_{\theta\theta} = r^2 \\ ds^2 = dr^2 + r^2 d\theta^2 \end{aligned} \tag{2.80}$$

Cartesian coordinates in the (flat) Minkowski space-time :

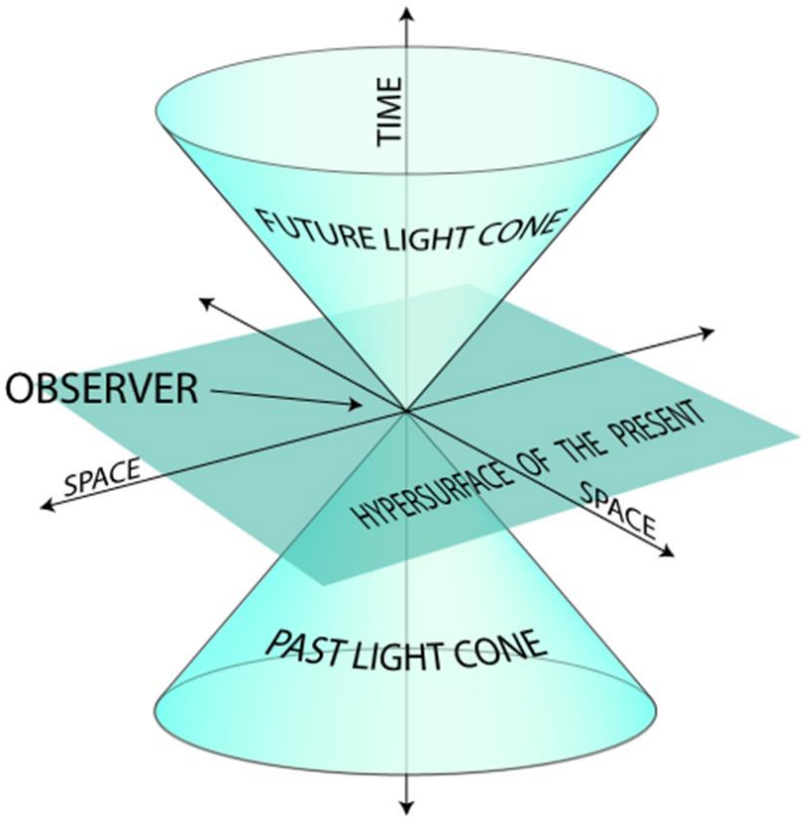
$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 \tag{2.81}$$

In an arbitrary curved space, an orthonormal basis can be adopted in any point. If $\vec{e}_{\hat{t}}$ is tangent vector to the world line of an observer, then $\vec{e}_{\hat{t}} = \vec{u}$ where $\vec{u}$ is the 4-velocity of the observer. In this case, we are using what we call the *comoving orthonormal basis* of the observer. In a such basis, we have the Minkowski-metric:

$$ds^2 = \eta_{\hat{\mu}\hat{\nu}}dx^{\hat{\mu}}dx^{\hat{\nu}} \tag{2.82}$$

The causal structure of spacetime

The causal structure of spacetime can be illustrated by considering the light cone.



The world lines of material particles or an observer, moving slower than light, are inside the light cone. Such curves are called *time-like*. The invariant parameter of a time-like curve is usually chosen to be the proper time τ of an observer following the curve. Then a tangent vector of the curve is the 4-velocity of the observer.

A point in spacetime represents an event. The distance in spacetime between two infinitesimally nearby points in spacetime is called an *interval*. A time-like interval is the interval between two pints on a time-like curve. It has $ds_{time}^2 < 0$.

We shall now give a general physical interpretation of the line element for time-like intervals. For this purpose it is sufficient to consider the Minkowski line-element which can be written

$$ds^2 = -c^2 \left\{ 1 - \frac{1}{c^2} \left[\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 + \left(\frac{dz}{dt} \right)^2 \right] \right\} dt^2$$

Consider a particle moving with a coordinate velocity

$$\vec{v} = v^x \vec{e}_x + v^y \vec{e}_y + v^z \vec{e}_z = \frac{dx}{dt}\vec{e}_x + \frac{dy}{dt}\vec{e}_y + \frac{dz}{dt}\vec{e}_z$$

Then

$$ds^2 = -\left[1 - \frac{(v^x)^2 + (v^y)^2 + (v^z)^2}{c^2} \right] c^2\, dt^2 = -\left(1 - \frac{v^2}{c^2} \right) c^2\, dt^2$$

From the special theory of relativity we know that the time measured by a standard clock following the particle, i.e. the *proper time* of the particle, is

$$d\tau = \sqrt{1 - \frac{v^2}{c^2}}\, dt \ .$$

Hence we obtain *the general physical interpretation of the line-element for a time-like interval*

$$ds^2 = -c^2 d\tau^2 \ .$$

This means that a time-like interval in space-time is measured by a clock, it is essentially a time interval.

We shall later define geodesic curves as the straightest possible curves between two events in spacetime. In flat Minkowski spacetime they are straight. We shall later show that geodesic curves have extremal length between two events.

Let us consider a time-like interval between two events O and P as shown on the Figure.

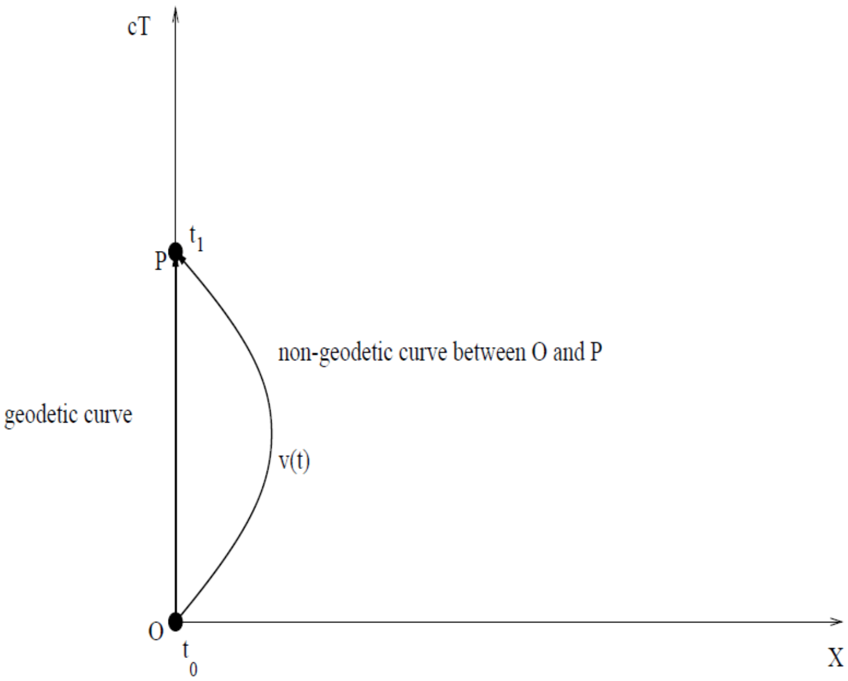


Figure 4.5: Timelike curves in spacetime.

(We shall not use capital letters for the coordinates in the text.)

We chose a coordinates commoving with a reference where an observer following a geodesic curve is at rest. The world line of the observer is the straight line from O to P. A particle also passing through the events O and P have accelerated motion and follows the curved world line on the figure. The particle has a non-vanishing velocity. Then we have for the proper time intervals measured by clocks following the observer and the particle, i. e. following the geodetic curve between the events and a non-geodesic curve,

$$\tau^2_{OPgeodetic} = (t_P - t_O)^2 > \left[\int_{t_O}^{t_P} \sqrt{1 - \frac{v^2}{c^2}} dt \right]^2 = \tau^2_{OPnon-geodetic} .$$

The interval between two events is path-dependent. The interval between to events has a *maximal* magnitude along the geodesic curve between the events.

For light the velocity is $v = c$. Then the proper time vanishes. The world line of light moving freely is called *light-like*, and an interval along the world line of light is called light-like. Hence *the interval along a light-like curve vanishes*, $ds^2_{light} = 0$. It is therefore also called a zero-interval.

A *spacelike* curve represents the world line of a particle moving faster than light. The interval between two events on such a curve is called a space-like interval and has $ds^2_{space} > 0$.