Lecture 9 12.02.18

3.1.1 Space geometry

Let $\vec{e}_{\hat{0}}$ be the 4-velocity field ($x^0 = ct, c = 1, x^0 = t$) of the reference particles in a reference frame R. A set of simultanous events in R, defines a 3 dimensional space called '3-space' in R. This space is orthogonal to $\vec{e}_{\hat{0}}$. We are going to find the metric tensor γ_{ij} in this space, expressed by the metric tensor $g_{\mu\nu}$ of spacetime.

In an arbitrary coordinate basis $\{\vec{e}_\mu\}$, $\{\vec{e}_i\}$ is not necessarily orthogonal to $\vec{e}_0$. We choose $\vec{e}_0 \| \vec{e}_{\hat{0}}$. Let $\vec{e}_{\perp i}$ be the component of $\vec{e}_i$ orthogonal to $\vec{e}_0$, that is:$\vec{e}_{\perp i} \cdot \vec{e}_0 = 0$. The metric tensor of space is defined by:

$$\gamma_{ij} = \vec{e}_{\perp i} \cdot \vec{e}_{\perp j}, \gamma_{i0} = 0, \gamma_{00} = 0$$

$$\vec{e}_{\perp i} = \vec{e}_i - \vec{e}_{\| i}$$

$$\vec{e}_{\| i} = \frac{\vec{e}_i \cdot \vec{e}_0}{\vec{e}_0 \cdot \vec{e}_o} \vec{e}_0 = \frac{g_{i0}}{g_{00}} \vec{e}_0$$

$$\begin{aligned}
\gamma_{ij} &= (\vec{e}_i - \vec{e}_{\| i}) \cdot (\vec{e}_j - \vec{e}_{\| j}) \\
&= (\vec{e}_i - \frac{g_{i0}}{g_{00}} \vec{e}_0) \cdot (\vec{e}_j - \frac{g_{j0}}{g_{00}} \vec{e}_0) \\
&= \vec{e}_i \cdot \vec{e}_j - \frac{g_{j0}}{g_{00}} \vec{e}_0 \cdot \vec{e}_i - \frac{g_{i0}}{g_{00}} \vec{e}_0 \cdot \vec{e}_j + \frac{g_{i0} g_{j0}}{g_{00}^2} \vec{e}_0 \cdot \vec{e}_0 \\
&= g_{ij} - \frac{g_{i0} g_{j0}}{g_{00}} - \frac{g_{i0} g_{j0}}{g_{00}} + \frac{g_{i0} g_{j0}}{g_{00}}
\end{aligned}$$

$$\Rightarrow \gamma_{ij} = g_{ij} - \frac{g_{i0} g_{j0}}{g_{00}} \tag{3.1}$$

(Note:$g_{ij} = g_{ji} \Rightarrow \gamma_{ij} = \gamma_{ji}$)
The line element in space:

$$dl^2 = \gamma_{ij} dx^i dx^j = \big(g_{ij} - \frac{g_{i0} g_{j0}}{g_{00}}\big) dx^i dx^j \tag{3.2}$$

gives the geometry of a "simultaneity space" in a reference frame where the metric tensor of spacetime in a comoving coordinate system is $g_{\mu\nu}$.

The line element for spacetime can be expressed as:

$$ds^2 = -d\hat{t}^2 + dl^2 \tag{3.3}$$

It follows that $d\hat{t} = 0$ represents the simultaneity defining the 3-space with metric γ_{ij}.

$$\begin{aligned}
d\hat{t}^2 &= dl^2 - ds^2 = (\gamma_{\mu\nu} - g_{\mu\nu})dx^\mu dx^\nu \\
&= (\gamma_{ij} - g_{ij})dx^i dx^j + 2(\gamma_{i0} - g_{i0})dx^i dx^0 + (\gamma_{00} - g_{00})dx^0 dx^0 \\
&= (g_{ij} - \frac{g_{i0}g_{j0}}{g_{00}} - g_{ij})dx^i dx^j - 2g_{i0}dx^i dx^0 - g_{00}(dx^0)^2 \\
&= -g_{00}\left[(dx^0)^2 + 2\frac{g_{i0}}{g_{00}}dx^0 dx^i + \frac{g_{i0}g_{j0}}{g_{00}^2}dx^i dx^j\right] \\
&= \left[(-g_{00})^{1/2}(dx^0 + \frac{g_{i0}}{g_{00}}dx^i)\right]^2
\end{aligned}$$

So finally we get

$$d\hat{t} = (-g_{00})^{1/2}(dx^0 + \frac{g_{i0}}{g_{00}}dx^i) \tag{3.4}$$

The 3-space orthogonal to the world lines of the reference particles in R, $d\hat{t} = 0$, corresponds to a coordinate time interval $dt = -\frac{g_{i0}}{g_{00}}dx^i$. This is not an exact differential , that is , the line integral of dt around a closed curve is in general not equal to 0. Hence you can not in general define simultaneity (given by $d\hat{t} = 0$) around closed curves. It can, however, be done if the spacetime metric is diagonal, $g_{i0} = 0$. The condition $d\hat{t} = 0$ means simultaneity on Einstein synchronized clocks . **Conclusion:It is in general ($g_{i0} \neq 0$) not possible to Einstein synchronize clocks around closed curves.**

Rotating reference frame

Let F be an

inertial frame with cylinder coordinates (T, R, Θ, Z). The line element is then given by

$$ds^2 = -dT^2 + dR^2 + R^2 d\Theta^2 + dZ^2 \quad (c = 1) \tag{3.9}$$

In a rotating reference frame, RF, we have cylinder coordinates (t, r, θ, z). We then have the following coordinate transformation :

$$t = T, \quad r = R, \quad \theta = \Theta - \omega T, \quad z = Z \tag{3.10}$$

The line element in the co-moving coordinate system in RF is then

$$\begin{aligned} ds^2 &= -dt^2 + dr^2 + r^2(d\theta + \omega dt)^2 + dz^2 \\ &= -(1 - r^2\omega^2)dt^2 + dr^2 + r^2 d\theta^2 + dz^2 + 2r^2\omega d\theta dt \quad (c = 1) \end{aligned} \tag{3.11}$$

The metric tensor have the following components:

$$\begin{aligned} &g_{tt} = -(1 - r^2\omega^2),\ g_{rr} = 1,\ g_{\theta\theta} = r^2,\ g_{zz} = 1 \\ &g_{\theta t} = g_{t\theta} = r^2\omega \end{aligned} \tag{3.12}$$

$dt = 0$ gives

$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2 \tag{3.13}$$

This represents the Euclidean geometry of the 3-space (simultaneity space, $t = T$) in IF.

The spatial geometry in the rotating system is given by the spatial line element:

$$\begin{aligned} dl^2 &= (g_{ij} - \frac{g_{i0}g_{j0}}{g_{00}})dx^i dx^j \\ \gamma_{rr} &= g_{rr} = 1,\ \gamma_{zz} = g_{zz} = 1, \\ \gamma_{\theta\theta} &= g_{\theta\theta} - \frac{g_{\theta 0}^2}{g_{00}} \\ &= r^2 - \frac{(r^2\omega)^2}{-(1 - r^2\omega^2)} = \frac{r^2}{1 - r^2\omega^2} \end{aligned}$$

Inserting this into the expression above for the spatial line element we have

$$dl^2 = dr^2 + \frac{r^2}{1-\frac{r^2\omega^2}{c^2}} d\theta^2 + dz^2 \ .$$

So we have a non Euclidean spatial geometry in RF. The circumference of a circle with radius r is

$$l_\theta = \frac{2\pi r}{\sqrt{1 - r^2\omega^2}} > 2\pi r \qquad (3.15)$$

We see that the quotient between circumference and radius $> 2\pi$ which means that the spatial geometry is *hyperbolic*. (For spherical geometry we have $l_\theta < 2\pi r$.)

We shall now try to explain this result first from the point of view of observers at rest in the non-rotating frame F, and then from the point of view of observes co-moving with the 'rotating' frame, R.

We shall first define the concept *standard measuring rod.* A standard measuring rod has by definition a constant rest length even if it is accelerated. It is not allowed by a standard measuring rod to be compressed or strained. Hence a standard measuring rod will have a Lorentz contraction according to its velocity.

As observed from F the measuring rods along a circle about the origin have a velocity $v = r\omega$. Hence they will be Lorentz contracted by the factor $\sqrt{1 - r^2\omega^2 / c^2}$. Hence there is place for more standard measuring rods around the circle the faster the frame R rotates. Therefore the measured length of the circle will be larger by this factor. This is the reason for the result (3.15) from the point of view of an F-observer. Hence according to the F-observers there is no question of a non-Euclidean geometry. The result (3.15) is explained by the Lorentz contraction of the standard measuring rods.

It may further be noted that since the material of a rotating disc cannot Lorentz contract an engraved scale on the disc cannot be used as a set of standard measuring rods. When the disc is put into rotation the material tries to Lorentz contract in the tangential direction, but is not allowed to do so. Hence a tangential strain will develop in the material of a disc that is put into rotation.

We shall now assume the validity of the principle of relativity for rotating motion. Then the observers in R can think of themselves as at rest and the environment as rotating. From this point of view the standard measuring rods are not Lorentz contracted. Hence the explanation of the F-observers does not work for the R-observers.

According to Einstein's interpretation of the general theory of relativity the explanation of the R-observes is as follows. The R-observer experiences what in Newton's theory is called a centrifugal force field. According the principle of equivalence this is reckoned as a gravitational field in the theory of relativity. The R-observer will say that there is a non-Euclidean spatial geometry in the R-frame, and that this is connected with the gravitational field which is present in this frame.

However, the experience of a gravitational acceleration field *locally* (the Newtonian centrifugal field) is due to the fact that the R-observers are at rest in a reference frame in which the reference particles are not freely falling.

It should be noted that in general an experimental result – in the present case that the measured length of a rotating disc with radius r is larger than $2\pi r$ – is independent of the reference frame that the experiment is described from, but the *explanation* of the result depends upon the motion of the observer's reference frame.