Lecture 33. 22. May 2018

Cosmology

10.1 Comoving coordinate system

We will consider expanding homogenous and isotropic models of the universe. We introduce an expanding frame of reference with the galactic clusters as reference particles. Then we introduce a 'comoving coordinate system' in this frame of reference with spatial coordinates χ, θ, ϕ. We use time measured on standard clocks carried by the galactic clusters as coordinate time (cosmic time). The line element can then be written in the form:

$$ds^2 = -dt^2 + a(t)^2[d\chi^2 + r(\chi)^2 d\Omega^2] \tag{10.1}$$

(For standard clocks at rest in the expanding system, $d\chi = d\Omega = 0$ and $ds^2 = -d\tau^2 = -dt^2$). The function $a(t)$ is called the expansion factor, and t is called cosmic time.

The physical distance, at a point of time t, to a particle with an instantaneous coordinate distance χ from an observer at the origin, is

$$l = a(t)\chi .$$

The velocity of the particle relative to the observer is

$$v = \dot{a}\chi + a\dot{\chi} = \frac{\dot{a}}{a}a\chi + a\dot{\chi} .$$

Defining the Hubble parameter

$$H = \frac{\dot{a}}{a},$$

this may be written as

$$v = Hl + a\dot{\chi} = v_H + v_P .$$

Here

$$v_H = Hl$$

is the velocity of the *Hubble flow*, which represent the expansion of the universe. It says that *the velocity of the Hubble flow is proportional to the distance from the observer*. This is the *Hubble law*. The general relativistic interpretation of this law is that *space expands*.

Furthermore

$$v_P = a\dot{\chi}$$

is called the *peculiar velocity* of the considered particle. It is a velocity peculiar to the considered particle due to a local gravitational particle at the particle. In other words: v_H *is the velocity of space, and* v_P *is a velocity through space*. The velocity of space is permitted to be larger than the velocity of light, which is the case farther away from the observer than c/H, and the velocity though space is always smaller than the velocity of light.

We shall now investigate whether the reference particles are freely moving, i.e. whether they obey the geodesic equation. Since a reference particle is permanently at rest in the coordinate system, and the derivative of the coordinate time with respect to its proper time is $dt/d\tau = 1$, it 4-velocity has components

$$u^\mu = \frac{dx^\mu}{d\tau} = \frac{dx^\mu}{dt} = (1,0,0,0) \tag{10.3}$$

This applies at an abritrary time, that is $\frac{du^\mu}{dt} = 0$. Geodesic equation: $\frac{du^\mu}{dt} + \Gamma^\mu{}_{\alpha\beta} u^\alpha u^\beta = 0$ which is reduces to: $\Gamma^\mu{}_{tt} = 0$

$$\Gamma^\mu{}_{tt} = \frac{1}{2} g^{\mu\nu} (\overbrace{g_{\nu t,t}}^{0} + \overbrace{g_{t\nu,t}}^{0} + \overbrace{g_{tt,\nu}}^{0}) = 0 \tag{10.4}$$

We have that $g_{tt} = -1$. This shows that the reference particles are freely falling.

10.2 Curvature isotropy - the Robertson-Walker metric

Introduce orthonormal form-basis:

$$\begin{aligned} &\underline{\omega}^{\hat{t}} = \underline{d}t \quad \underline{\omega}^{\hat{\chi}} = a(t)\underline{d}\chi \quad \underline{\omega}^{\hat{\theta}} = a(t)r(\chi)\underline{d}\theta \\ &\underline{\omega}^{\hat{\phi}} = a(t)r(\chi)\sin\theta \underline{d}\phi \end{aligned} \tag{10.5}$$

Using Cartans 1st equation:

$$\underline{d\omega}^{\hat{\mu}} = -\underline{\Omega}^{\hat{\mu}}{}_{\hat{\nu}} \wedge \underline{\omega}^{\hat{\nu}} \tag{10.6}$$

to find the connection forms. Then using Cartans 2nd structure equation to calculate the curvature forms:

$$\underline{R}^{\hat{\mu}}{}_{\hat{\nu}} = \underline{d\Omega}^{\hat{\mu}}{}_{\hat{\nu}} + \underline{\Omega}^{\hat{\mu}}{}_{\hat{\lambda}} \wedge \underline{\Omega}^{\hat{\lambda}}{}_{\hat{\nu}} \tag{10.7}$$

Calculations give: (notation: $\dot{} = \frac{d}{dt}$, $' = \frac{d}{d\chi}$)

$$\underline{R}^{\hat{t}}_{\ \hat{i}} = \frac{\ddot{a}}{a}\underline{\omega}^{\hat{t}} \wedge \underline{\omega}^{\hat{i}}, \qquad \underline{\omega}^{\hat{i}} = \underline{\omega}^{\hat{\chi}}, \underline{\omega}^{\theta}, \underline{\omega}^{\phi}$$

$$\underline{R}^{\hat{\chi}}_{\ \hat{j}} = (\frac{\dot{a}^2}{a^2} - \frac{r''}{ra^2})\underline{\omega}^{\hat{\chi}} \wedge \underline{\omega}^{\hat{j}}, \qquad \underline{\omega}^{\hat{j}} = \underline{\omega}^{\hat{\theta}}, \underline{\omega}^{\hat{\phi}} \tag{10.8}$$

$$\underline{R}^{\hat{\theta}}_{\ \hat{\phi}} = (\frac{\dot{a}^2}{a^2} + \frac{1}{r^2a^2} - \frac{r'^2}{r^2a^2})\underline{\omega}^{\hat{\theta}} \wedge \underline{\omega}^{\hat{\phi}}$$

The curvature of 3-space ($dt = 0$) can be found by putting $a = 1$. That is:

$$_3\underline{R}^{\hat{\chi}}_{\ \hat{j}} = -\frac{r''}{r}\underline{\omega}^{\hat{\chi}} \wedge \underline{\omega}^{\hat{j}}$$

$$_3\underline{R}^{\hat{\theta}}_{\ \hat{\phi}} = (\frac{1}{r^2} - \frac{r'^2}{r^2})\underline{\omega}^{\hat{\theta}} \wedge \underline{\omega}^{\hat{\phi}} \tag{10.9}$$

The 3-space is assumed to be isotropic and homogenous. This demands

$$-\frac{r''}{r} = \frac{1 - r'^2}{r^2} = k \ , \tag{10.10}$$

where k represents the constant curvature of the 3-space.

$$\therefore \ r'' + kr = 0 \quad \text{and} \quad r' = \sqrt{1 - kr^2} \tag{10.11}$$

$$\begin{aligned} \sqrt{-k}r &= \sinh(\sqrt{-k}\chi) \qquad (k < 0) \\ r &= \chi \qquad (k = 0) \\ \sqrt{k}r &= \sin(\sqrt{k}\chi) \qquad (k > 0) \end{aligned} \tag{10.12}$$

The solutions can be characterized by the following 3 cases:

$$\begin{aligned} r &= \sinh\chi, \qquad dr = \sqrt{1 + r^2}d\chi, \qquad (k = -1) \\ r &= \chi, \qquad dr = d\chi, \qquad (k = 0) \\ r &= \sin\chi, \qquad dr = \sqrt{1 - r^2}d\chi, \qquad (k = 1) \end{aligned} \tag{10.13}$$

In all three cases one may write $dr = \sqrt{1 - kr^2}d\chi$, which is just the last equation above.

We now set $d\chi^2 = \frac{dr^2}{1-kr^2}$ into the line-element :

$$\begin{aligned} ds^2 &= -dt^2 + a^2(t)\left(d\chi^2 + r^2(\chi)d\Omega^2\right) \\ &= -dt^2 + a^2(t)\left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2\right) \end{aligned} \tag{10.14}$$

The first expression is known as the standard form of the line-element, the second is called the Robertson-Walker line-element.

The 3-space has constant curvature. 3-space is spherical for $k = 1$, Euclidean for $k = 0$ and hyperbolic for $k = -1$.

Universe models with $k = 1$ are known as 'closed' and models with $k = -1$ are known as 'open'. Models with $k = 0$ are called 'flat' even though these models also have curved space-time.