Differential forms

40.1.1. Definition. The EXTERIOR DIFFERENTIATION OPERATOR d is a mapping which takes k-forms to $(k+1)$-forms. That is $d\colon G^k \longrightarrow G^{k+1}$. It has the following properties:

(i) If f is a 0-form on $\mathbb{R}^3$, then $d(f)$ is just the differential (or total derivative) df of f. That is,

$$d(f) = df = f_x\,dx + f_y\,dy + f_z\,dz;$$

(ii) d is linear (that is, if ω and μ are k-forms and c is a constant, then $d(\omega + \mu) = d\omega + d\mu$ and $d(c\omega) = c\,d\omega$);

(iii) $d^2 = \mathbf{0}$ (that is, $d(d\omega) = \mathbf{0}$ for every k-form ω);

(iv) If ω is a k-form and μ is any differential form

$$d(\omega \wedge \mu) = (d\omega) \wedge \mu + (-1)^k \omega \wedge d\mu.$$

Example: If $\omega = xy^2 \sin z$, then

$$d\omega = y^2 \sin z\,dx + 2xy \sin z\,dy + xy^2 \cos z\,dz \qquad \text{[by (i)]}$$

Example: If $\omega = x^2 e^z$, then

$$d\omega = 2xe^z\,dx + x^2 e^z\,dz \qquad \text{[by (i)]}$$

Example: If $\omega = xy^2 z^3\,dy$, then

$$\begin{aligned}
d\omega &= d(xy^2z^3) \wedge dy + xy^2z^3\,d(dy) && \text{[by (iv)]}\\
&= d(xy^2z^3) \wedge dy && \text{[by (iii)]}\\
&= (y^2z^3\,dx + 2xyz^3\,dy + 3xy^2z^2\,dz) \wedge dy && \text{[by (i)]}\\
&= y^2z^3\,dx \wedge dy + 2xyz^3\,dy \wedge dy + 3xy^2z^2\,dz \wedge dy\\
&= y^2z^3\,dx \wedge dy - 3xy^2z^2\,dy \wedge dz
\end{aligned}$$

Example: If $\omega = xy\,dx + x^2yz\,dy + z^3\,dz$, then

$$\begin{aligned}
d\omega &= d(xy\,dx) + d(x^2yz\,dy) + d(z^3\,dz) && \text{[by (ii)]}\\
&= d(xy) \wedge dx + xy\,d(dx) + d(x^2yz) \wedge dy + x^2yz\,d(dy)\\
&\qquad + d(z^3) \wedge dz + z^3\,d(dz) && \text{[by (iv)]}\\
&= d(xy) \wedge dx + d(x^2yz) \wedge dy + d(z^3) \wedge dz && \text{[by (iii)]}\\
&= (y\,dx + x\,dy) \wedge dx + (2xyz\,dx + x^2z\,dy + x^2y\,dz) \wedge dy\\
&\qquad + (3z^2\,dz) \wedge dz && \text{[by (i)]}\\
&= x\,dy \wedge dx + 2xyz\,dx \wedge dy + x^2y\,dz \wedge dy\\
&= x(2yz - 1)\,dx \wedge dy - x^2y\,dy \wedge dz.
\end{aligned}$$

Remark: It simplifies computations to notice that $d(dx \wedge dy) = d(dx) \wedge dy - dx \wedge d(dy) = \mathbf{0} \wedge dy - dx \wedge \mathbf{0} = \mathbf{0}$.

Example: If $\omega = xz^2\,dx \wedge dy + xyz\,dz \wedge dx$, then

$$\begin{aligned}
d\omega &= d(xz^2\,dx \wedge dy) + d(xyz\,dz \wedge dx) && \text{[by (ii)]}\\
&= d(xz^2) \wedge (dx \wedge dy) + xz^2\,d(dx \wedge dy) + d(xyz) \wedge (dz \wedge dx)\\
&\qquad + xyz\,d(dz \wedge dx) && \text{[by (iv)]}\\
&= d(xz^2) \wedge (dx \wedge dy)\\
&\qquad + d(xyz) \wedge (dz \wedge dx) && \text{[by the Remark above]}\\
&= (z^2\,dx + 2xz\,dz) \wedge dx \wedge dy\\
&\qquad + (yz\,dx + xz\,dy + xy\,dz) \wedge dz \wedge dx && \text{[by (i)]}\\
&= 2xz\,dz \wedge dx \wedge dy + xz\,dy \wedge dz \wedge dx\\
&= -2xz\,dx \wedge dz \wedge dy - xz\,dy \wedge dx \wedge dz\\
&= 2xz\,dx \wedge dy \wedge dz + xz\,dx \wedge dy \wedge dz\\
&= 3xz\,dx \wedge dy \wedge dz
\end{aligned}$$

Example: If $\omega = xy^2z^3\,dx \wedge dy \wedge dz$, then $d\omega = \mathbf{0}$. [**Proof:** The differentiation operator takes 3-forms to 4-forms and (in this course) all 4-forms are zero. Or, you can give essentially the same argument as in the Remark above.]

40.2. Exercises

(1) Let $f(x, y, z) = x^3 + y^2 + 2x \sin z$. Then
$$df = \_\_\_\_\_\_\_\_\_\_ dx + \_\_\_\_\_\_ dy + \_\_\_\_\_\_\_\_ dz\,.$$

(2) If $\omega = x^3y^2z^5\,dy$, then
$$d\omega = \_\_\_\_\_\_\_\_\_\_ dy \wedge dz + \_\_\_\_\_\_ dz \wedge dx + \_\_\_\_\_\_\_\_ dx \wedge dy.$$

(3) Let $\omega = \cos(xy^2)\,dx \wedge dz$. Then (in simplified form)
$$d\omega = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ .$$

(4) If $\omega = xy^2z^3\,dx + \sin(xy)\,dy + e^{yz}\,dz$, then
$$d\omega = \_\_\_\_\_\_\_\_ dy \wedge dz + \_\_\_\_\_\_ dz \wedge dx + \_\_\_\_\_\_\_\_\_\_ dx \wedge dy.$$

(5) Let $\omega = x^2y\,dy - xy^2\,dx$. Then (in simplified form)
$$d\omega = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ .$$

(6) Let $\omega = x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy$. Then (in simplified form)
$$d\omega = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ .$$

(7) Let f be a 0-form. Then (in simplified form)
$$d(f\,dx) = \_\_\_\_ dy \wedge dz + \_\_\_\_ dz \wedge dx + \_\_\_\_ dx \wedge dy.$$

(8) Let $\omega = 3xz\,dx + xy^2\,dy$ and $\mu = x^2y\,dx - 6xy\,dz$. Then (in simplified form)
$$d(\omega \wedge \mu) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ .$$

(9) Let $\omega = yz\,dx + xz\,dy + xy\,dz$. Then
$$d\omega = \_\_\_\_ dy \wedge dz + \_\_\_\_ dz \wedge dx + \_\_\_\_ dx \wedge dy.$$

(10) Let $\omega = 2x^5e^z\,dx + y^3\sin z\,dy + (x^2 + y)\,dz$. Then
$$d\omega = \_\_\_\_\_\_\_\_\_\_ dy \wedge dz + \_\_\_\_\_\_\_\_\_\_ dz \wedge dx + \_\_\_\_\_\_ dx \wedge dy.$$

(11) Let $\omega = x\,dx + xy\,dy + xyz\,dz$. Then
$$d\omega = \_\_\_\_\_\_\_\_ dy \wedge dz + \_\_\_\_\_\_\_\_ dz \wedge dx + \_\_\_\_\_\_\_\_ dx \wedge dy.$$

40.4. Answers to Odd-Numbered Exercises

(1) $3x^2 + 2\sin z,\ 2y,\ 2x\cos z$

(3) $2xy\sin(xy^2)\,dx \wedge dy \wedge dz$

(5) $4xy\,dx \wedge dy$

(7) $0,\ f_3,\ -f_2$

(9) $0,\ 0,\ 0$

(11) $xz,\ -yz,\ y$

Homework 3

1. Prove that the 1-form $x\,dy - y\,dx$ is invariant under all rotations of $\mathbb{R}^2$ around $(0,0)$.

2. Prove the same for $x\,dx + y\,dy$.

3. Introduce "hyperbolic coordinates" (χ, ψ): $x = \chi\cosh\psi$, $y = \chi\sinh\psi$. In which domain $U \subset \mathbb{R}^2$ are they defined?

4. Calculate the area 2-form $dx \wedge dy$ of $\mathbb{R}^2$ in the polar coordinates (ρ, φ): $dx \wedge dy = C\,d\rho \wedge d\varphi$, $C = ?$.

5. Calculate the area 2-form $dx \wedge dy$ in the complex coordinates z, $\overline{z}$, $dx \wedge dy = C\,dz \wedge d\overline{z}$, $C = ?$.

6. Calculate the volume 3-form of $\mathbb{R}^3$ in the spherical coordinates (r, φ, θ),

$$\begin{aligned} z &= r\cos\theta, \\ x &= r\sin\theta\cos\varphi, \\ y &= r\sin\theta\sin\varphi, \end{aligned}$$

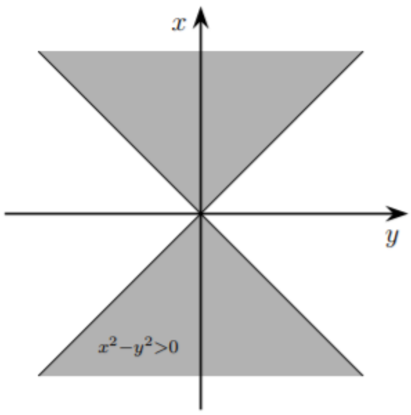
$\underbrace{dx \wedge dy \wedge dz}_{\text{the volume 3-form}} = (?)\,dr \wedge d\theta \wedge d\varphi.$

Homework 3 — Solutions

1. $x\,dy - y\,dx = d\varphi \cdot \rho^2$ — invariant under rotation.

2. $x\,dx + y\,dy = \rho\,d\rho$ — invariant under rotation: $\rho \mapsto \rho$, $\varphi \mapsto \varphi + \text{const}$, $d\rho \mapsto d\rho$, $d\varphi \mapsto d\varphi$.

   $x = \chi\cosh\psi$, $y = \chi\sinh\psi$, $x^2 - y^2 = \chi^2 > 0$



3. $dx \wedge dy = \rho\,d\rho \wedge d\varphi$.

4.
$$dx \wedge dy = \frac{dz \wedge d\overline{z}}{-2i}, \qquad \begin{cases} dz = dx + i\,dy, \\ d\overline{z} = dx - i\,dy. \end{cases}$$

5. $\mathbb{R}^3$: $dx \wedge dy \wedge dz = r^2 \sin\theta\,dr \wedge d\theta \wedge d\varphi$.

   For the unit sphere S^2 $(r = 1)$ we have Area $= \sin\theta\,d\theta \wedge d\varphi$, $0 \leqslant \theta \leqslant \pi$.

Example 1.1. Express the 2-form $dx \wedge dy$ in polar coordinates.

Solution: From $dx = \cos\theta\, dr - r\sin\theta\, d\theta$ and $dy = \sin\theta\, dr + r\cos\theta\, d\theta$, we have

$$\begin{aligned} dx \wedge dy &= \cos\theta . r\cos\theta\ dr \wedge d\theta - r\sin\theta .\ \sin\theta\ d\theta \wedge dr \\ &= r\cos^2\theta dr \wedge d\theta + r\sin^2\theta dr \wedge \theta = rdr \wedge d\theta, \end{aligned}$$

by noticing that $dr \wedge dr = 0$ and $d\theta \wedge d\theta = 0$.

[Aside: we may also apply the identities $rdr = xdx + ydy$ and $d\theta = (xdy - ydx)/(x^2 + y^2)$ to obtain

$$rdr \wedge d\theta = (xdx + ydy) \wedge \frac{xdy - ydx}{x^2 + y^2} = \frac{x^2dx \wedge dy - y^2dy \wedge dx}{x^2 + y^2} = dx \wedge dy. \quad \Big]$$

Example 1.3. Consider a mapping f from $\mathbf{R}^2$ to $\mathbf{R}^2$ sending (x, y) to (u, v), where $u = x^2 - y^2$ and $v = 2xy$. Express $du \wedge dv$ in terms of $dx \wedge dy$.

Solution: We have $du = 2xdx - 2ydy$ and $dv = 2xdy + 2ydx$. Hence

$$\begin{aligned} du \wedge dv &= (2xdx - 2ydy) \wedge (2xdy + 2ydx) \\ &= 4x^2dx \wedge dy - 4y^2dy \wedge dx = 4(x^2 + y^2)\, dx \wedge dy. \end{aligned}$$

Example 1.4. Recall our favorite example: $\omega = \dfrac{xdy - ydx}{x^2 + y^2}$, the angular form. Find its exterior derivative $d\omega$.

Solution: Rewrite the identity $\omega = (xdy - ydx)/(x^2 + y^2)$ as $(x^2 + y^2)\omega = xdy - ydx$. Take exterior derivatives of both sides. Using the product rule, the left hand side gives

$$\begin{aligned} d((x^2 + y^2)\omega) &= d(x^2 + y^2) \wedge \omega + (x^2 + y^2)d\omega \\ &= 2(xdx + ydy) \wedge (xdy - ydx)/(x^2 + y^2) + (x^2 + y^2)d\omega. \end{aligned}$$

Here the expression $(xdx + ydy) \wedge (xdy - ydx)$ can be expanded as

$$x^2dx \wedge dy - y^2dy \wedge dx = x^2dx \wedge dy + y^2dx \wedge dy = (x^2 + y^2)dx \wedge dy$$

(we have seen this in Example 1.2) and hence $d((x^2 + y^2)\omega) = 2dx \wedge dy + (x^2 + y^2)d\omega$. On the other hand, $d(xdy - ydx) = dx \wedge dy - dy \wedge dx = 2dx \wedge dy$. Thus we have $2dx \wedge dy + (x^2 + y^2)d\omega = 2dx \wedge dy$. Cancelling $2dx \wedge dy$, we get $(x^2 + y^2)d\omega = 0$.

Example 1.5. The general 1-form in three variables x, y and z can be written as $\alpha = Pdx + Qdy + Rdz$, with P, Q, R functions of x, y, z. Find its exterior derivative $d\alpha$.

Solution: Using the identities $dP = P_x dx + P_y dy + P_z dz$ (recall that P_x stands for $\partial P/\partial x$), $dQ = Q_x dx + Q_y dy + Q_z dz$ and $dR = R_x dx + R_y dy + R_z dz$, we compute:

$$
\begin{aligned}
d\alpha &= dP \wedge dx + dQ \wedge dy + dR \wedge dz \\
&= (P_x dx + P_y dy + P_z dz) \wedge dx + (Q_x dx + Q_y dy + Q_z dz) \wedge dy + (R_x dx + R_y dy + R_z dz) \wedge dz \\
&= P_y dy \wedge dx + P_z dz \wedge dx + Q_x dx \wedge dy + Q_z dz \wedge dy + R_x dx \wedge dz + R_y dy \wedge dz \\
&= (R_y - Q_z)\, dy \wedge dz + (P_z - R_x)\, dz \wedge dx + (Q_x - P_y)\, dx \wedge dy.
\end{aligned}
$$

The reader should notice the connection between the exterior derivative $d\alpha$ and the curl of a vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ in classical vector analysis, which is defined by the identity

$$\text{curl}\, \mathbf{F} \equiv \nabla \times \mathbf{F} = (R_y - Q_z,\ P_z - R_x,\ Q_x - P_y).$$

Also, notice the connection between $df = f_x dx + f_y dy + f_z dz$ (here f is a function of three variables x, y and z) and the gradient of f defined by

$$\text{grad}\, f \equiv \nabla f = f_x \mathbf{i} + f_y \mathbf{j} + f_z \mathbf{k}.$$

Example 10: Let $\omega = (x_1 + x_3^2) dx_1 \wedge dx_2$. Then

$$
\begin{aligned}
d\omega &= d(x_1 + x_3^2) \wedge dx_1 \wedge dx_2 \\
&= dx_1 \wedge dx_1 \wedge dx_2 + 2x_3\, dx_3 \wedge dx_1 \wedge dx_2 \\
&= 2x_3\, dx_3 \wedge dx_1 \wedge dx_2 \\
&= -2x_3\, dx_1 \wedge dx_3 \wedge dx_2 \\
&= 2x_3\, dx_1 \wedge dx_2 \wedge dx_3
\end{aligned}
$$

Notice the term with two dx_1's is zero, and in the last two steps I just put $d\omega$ in "standard" form.

Problem 2.27 Consider on $\mathbb{R}^2$:

$$X = \left(x^2 + y\right)\frac{\partial}{\partial x} + \left(y^2 + 1\right)\frac{\partial}{\partial y}, \qquad Y = (y-1)\frac{\partial}{\partial x},$$
$$\theta = \left(2xy + x^2 + 1\right)\mathrm{d}x + \left(x^2 - y\right)\mathrm{d}y,$$

Compute:

(i) $[X, Y]_{(0,0)}$.
(ii) $\theta(X)(0, 0)$.

Solution

(i)

$$[X, Y] = \left(y^2 - 2xy + 2x + 1\right)\frac{\partial}{\partial x}, \quad \text{so} \quad [X, Y]_{(0,0)} = \left.\frac{\partial}{\partial x}\right|_{(0,0)}.$$

(ii)

$$\theta(X)(0,0) = \left(\left(2xy + x^2 + 1\right)\left(x^2 + y\right) + \left(x^2 - y\right)\left(y^2 + 1\right)\right)(0,0) = 0.$$