the frequency ν_0 is received by the observer in $r = R$, $\phi = 0$. Which frequency ν will be measured by the observer?

(d) We now assume that clocks are tightly packed around the disc $r = R$. The clocks are non-moving with respect to the disc, and are standard clocks, measuring proper time of the clock. We now want to synchronize the clocks and start out with a clock at the point $(R, 0)$. The clocks are then synchronized in the direction of increasing ϕ in the following way: When the clock is tuned at the point ϕ, the clock at the neighbouring point $\phi + d\phi$ is also tuned so that they show the same time at simultaneity in the instantaneous rest frame of the two clocks.
Show that there is a problem with synchronization when this process is performed around the entire circle, by the fact that the clock we started out with is no longer synchronous with the neighbouring clock which is tuned according to the synchronization process. Find the time difference between these two clocks.

(e) Locally around a point (r, ϕ, t) we can define an inertial system being an instantaneous rest frame of the point (r, ϕ) on the disc. We introduce an orthonormal set of basis vectors $\vec{e}_{\hat{\lambda}}$, $\vec{e}_{\hat{\eta}}$ and $\vec{e}_{\hat{\xi}}$ in this frame. The vector $\vec{e}_{\hat{\lambda}}$ points along the time axis of the system, $\vec{e}_{\hat{\xi}}$ points radially and $e_{\hat{\eta}}$ tangentially. Find the vectors expressed by $\vec{e}_t$, $\vec{e}_x$ and $\vec{e}_y$.
The path of a light signal from A is studied at the local inertial frames along the path. Find how the spatial direction of the light signal changes relative to the basis vectors $\vec{e}_{\hat{\xi}}$ and $\vec{e}_{\hat{\eta}}$ (outwards) along the paths. Is this result in accordance with what was earlier found on the path of the light in the $(\bar{x}, \bar{y})$-system?

4.2. *Free particle in a hyperbolic reference frame*
The metric for a 2-dimensional space is given by

$$ds^2 = -V^2 dU^2 + dV^2. \tag{4.56}$$

(a) Find the Euler–Lagrange equations for the motion of a free particle using this metric. Show that they admit the following solution:

$$\frac{1}{V} = \frac{1}{V_0} \cosh(U - U_0).$$

What is the physical interpretation of the constants V_0 and U_0?

(b) Show that these are straight lines in the coordinate system (t, x) given by

$$\begin{aligned} x &= V \cosh U, \\ t &= V \sinh U. \end{aligned} \tag{4.57}$$

Express the speed of the particle in terms of U_0, and its x-component at $t = 0$ in terms of V_0 and U_0. Find the interval ds^2 expressed in terms of x and t and show that the space in which the particle is moving is a Minkowski space with one time and one spatial dimension.

(c) Express the covariant component p_U of the momentum using $p_t = -E$ and $p_x = p$, and show that it is a constant of motion. How can this fact be directly extracted from the metric? Show further that the contravariant component p^U is not a constant of motion. Are p_V or p^V constants of motion?

4.3. *Uniformly accelerated system of reference*
We will now study a curved coordinate system (U,V) in a 2-dimensional Minkowski space. The connection with the Cartesian system (t,x) is given by

$$t = V\sinh(aU), \tag{4.58}$$

$$x = V\cosh(aU), \tag{4.59}$$

where a is a constant. (See Problem 4.2.)

(a) Draw the coordinate lines U and V in a (t,x)-diagram. Calculate the basis vectors $\vec{e}_U$ and $\vec{e}_V$ and draw them at some chosen points in the diagram. Find the metric $ds^2 = dx^2 - c^2dt^2$ expressed by U and V.
(b) We now assume that a particle has a path in spacetime so that it follows one of the curves $V =$ constant. Such a motion is called hyperbolic motion. Why?
Show that the particle has constant acceleration g along the path when the acceleration is measured in the instantaneous rest frame of the particle. Find the acceleration. Find also the velocity and accelereation of the particle in the stationary system (t,x).
(c) Show that at any point on the particle trajectory, the direction of the (U,V)-coordinate axis will overlap with the time and spatial axis of the instantaneous rest frame of the particle. Explain why it is possible to see from the line element that the V-coordinate measures length along the spatial axis, whereas the U-coordinate, which is the coordinate time, is in general not the proper time of the particle? For what value of V is the coordinate equal to the proper time?
The (U,V)-coordinate system can be considered as an attempt to construct, from the instantaneous rest frames along the path, a coordinate system covering the entire spacetime. Explain why this is not possible for the entire space. (Hint: There is a coordinate singularity at a certain distance from the trajectory of the particle.)
(d) A rod is moving in the direction of its own length. At the time $t = 0$ the rod is at rest, but still accelerated. The length of the rod measured in the stationary system is L at this time. The rod moves so that the forwards point of the rod has constant rest acceleration measured in the instantaneous rest frame.
We assume that the acceleration of the rod finds place so that the infinitesimal distance $d\ell$ between neighbouring points on the rod measured in the instantaneous rest frame are constant. Find the motion of the rear point of the rod in the stationary reference system. Why is there a maximal length of the rod, L_{max}?
If the rear point of the rod has constant acceleration and the rod is accelerated as previously in this exercise, then is there a maximal value of L?
(e) A spaceship leaves the Earth at the time $t = 0$ and moves with a constant acceleration g, equal to the gravitational constant at the Earth, into space. Find how far the ship has travelled during 10 years of proper time of the ship.

Radiosignals are sent from the Earth towards the spaceship. Show that signals that are sent after a given time T will never reach the ship (even if the signals travel with the speed of light). Find T. At what time are the signals sent from the Earth if they reach the ship after 10 years (proper time of the ship)?
Calculate the frequency of the radiosignals received by the ship, given by the frequency ν_0 (emitter frequency) and the time t_0 (emitter time). Investigate the behaviour of the frequency when $t_0 \to T$.

4.4. *The projection tensor*
Let the metric tensor of the spacetime in a coordinate system K have components $g_{\mu\nu}$. An observer has a 4-velocity given by $\vec{u}$.
An arbitrary vector $\vec{a}$ can be decomposed into a component $\vec{a}_{\parallel}$ parallel to $\vec{u}$ and a component $\vec{a}_{\perp}$ orthogonal to $\vec{u}$, so that $\vec{a} = \vec{a}_{\parallel} + \vec{a}_{\perp}$.

(a) Show that

$$\vec{a}_{\parallel} = (\vec{a}\cdot\vec{u})\vec{u}/u^2 = -(\vec{a}\cdot\vec{u})\vec{u}, \tag{4.60}$$

$$\vec{a}_{\perp} = \vec{a} + (\vec{a}\cdot\vec{u})\vec{u}. \tag{4.61}$$

Equation (4.61) can be rewritten by the *projection tensor*

$$P = \underline{I} + \vec{u}\otimes\underline{u}, \tag{4.62}$$

where $\underline{I}$ is the vectorial 1-form that can be written as

$$\underline{I} = \delta^{\mu}{}_{\nu}\vec{e}_{\mu}\otimes\underline{\omega}^{\nu} \tag{4.63}$$

and $\underline{u}$ has components $u_{\mu} = g_{\mu\nu}u^{\nu}$.
Show that Eq. (4.61) can be written as

$$\vec{a}_{\perp} = P(\ \ ,\vec{a}) \tag{4.64}$$

so that the components of $\vec{a}_{\perp}$ and $\vec{a}$ are related via

$$a^{\mu}_{\perp} = P^{\mu}{}_{\nu}a^{\nu}. \tag{4.65}$$

Since $\vec{u}$ is tangent vector of the world line of the observer, then $P^{\mu}{}_{\nu}a^{\nu}$ is the projection of $\vec{a}$ in the spatial plane of simultaneity orthogonal to the time vector in the local orthonormal basis of the observer.

(b) Assume that the observer is non-moving in K. Find the mixed and covariant components of P.

(c) Let $\vec{a}$ be the 4-acceleration of a particle. What kind of motion does the covariant equation $P^{\mu}{}_{\nu}\frac{da^{\nu}}{d\tau} = 0$ describe. Explain! (Hint: Find the time and space components of this equation. In an instantaneous rest frame of the particle, $d\vec{a}/d\tau = (g^2, d\vec{g}/d\tau)$ where the 3-vector $\vec{g}$ is the rest acceleration.)