UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Exam in: General Relativity (FYS4160)

Day of exam: June 16, 2022

Exam hours: 4 hours

This examination paper consists of 4 pages. (including the title page)

Appendices: none

Permitted materials: 3 A4 pages (two-sided) with own notes.

Make sure that your copy of this examination paper is complete before answering.

Final exam

Lecture spring 2022: General Relativity (FYS4160)

⇝ *Carefully **read all questions** before you start to answer them! Note that you do not have to answer the questions in the order presented here, so try to answer those first that you feel most sure about. Keep your descriptions as short and concise as possible! Answers given in English are preferred – but feel free to write in Norwegian if you struggle with formulations! Maximal number of available points: 50.*

Good luck !!!

Problem 1 *(11 points)*

a) State the equivalence principle in its strongest form, as formulated by Einstein when deriving the theory of general relativity. Argue why this principle strongly suggests to describe gravity as being geometric in nature! What is the decisive difference to other forces of nature, e.g. the electromagnetic force? *(3 points)*

b) State the equation of motion of a force-free test particle in special and general relativity, in arbitrary coordinates, and discuss the difference! How would external forces appear in these equations? *(3 points)*

c) Explain why a theory satisfying the equivalence principle must be formulated in terms of tensorial quantities. Show that the Kronecker symbol $\delta^\mu{}_\nu$ is a tensor (i.e. it describes the components of a tensor in *every* coordinate system), while the Levi-Civita symbol $\tilde{\epsilon}_{\mu\nu\rho\sigma}$ is not. What can be done to promote the latter to a tensorial quantity, and what is the interpretation (or most common application) of the resulting tensor? *(5 points)*

Problem 2 *(8 points)*
A spacetime is said to be of constant positive curvature if it satisfies the relation

$$R_{\mu\nu\rho\sigma} = \frac{1}{L^2}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right) . \qquad (1)$$

a) In n space-time dimensions, where $\delta^\alpha{}_\alpha = n$, how does the real constant L relate to the Ricci scalar R ? *(2 points)*

b) Express the Riemann and Ricci tensors explicitly in terms of R, rather than L! Would these expressions change for spacetimes with constant *negative* curvature? *(2 points)*

c) Show that such a spacetime with constant positive curvature solves Einstein's equations in vacuum for a cosmological constant $\Lambda > 0$, and state the value of L for which this is the case (again in n dimensions)! *(2 points)*

d) The so-called de-Sitter space that you looked at in the previous problem can be represented by the line element $ds^2 = dt^2 - \exp(Ct)\,(dr^2 + r^2 d\Omega)$. What is the value of the constant C? In cosmology, an early period of inflation – where the line-element was very close to that of de-Sitter space – is usually quoted as being decisive in guaranteeing that the early universe was flat (i.e. no curvature) to a very high degree (see also problem 4). Why is that no contradiction to the positive curvature that you determined in c) for this spacetime? *(2 points)*

<u>Problem 3</u> *(15 points)*

Consider a spacecraft hovering at a fixed point, located at a radial distance R away from the horizon of a Schwarzschild black hole.

a) Calculate the thrust, i.e. outward acceleration, that is needed to maintain the position of the spacecraft! How does this compare to the Newtonian result for a point-like object of the same mass? *(5 points)*

b) At some point the engine suddenly fails, and the spacecraft is sucked into the black hole. How much time passes on board of the spacecraft, from the time it passes the event horizon until it reaches the singularity? Briefly discuss the limiting cases of very small and very large R. *(6 points)*
[Hint: The definite integral $\int_0^\infty dx\,(1+x^2)^{-2} = \pi/4$ might come in handy...]

c) Assume that, instead, the crew quickly gets the engine to work again – before they reach the event horizon, but still not sufficiently fast to avoid passing the horizon. What is the rate $dr/d\tau$ by which the Schwarzschild coordinate r decreases *at least* inside the black hole, independently of the (radial) trajectory of the spacecraft? In order to maximize the time spent inside the event horizon, how should the crew control the thrust of the engine (from the moment they have control again)? For a black hole mass of about $M \sim 2 \cdot 10^8 M_\odot$ – similar in size to the supermassive black hole at the center of the Andromeda galaxy – how many minutes would it take *at most* to reach the singularity, after passing the horizon? *(4 points)*
[Hint: The Schwarzschild radius of the sun is 2.9 km.]

<u>Problem 4</u> *(16 points)*

In this problem we want to derive how free particles move in Friedman-Robertson-Waker spacetimes, where the line element can be written as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a^2(t)\left[\frac{dr^2}{1-\kappa r^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right]. \qquad (2)$$

a) What type of coordinates are used here? State this line element instead in Cartesian coordinates, using the fact that the spatial curvature of the universe is observationally determined to be very close to zero. Why can we conclude from this observation that spatial curvature can also be neglected at earlier times? *(3 points)*

b) Show that the only non-vanishing Christoffel symbols are given by $\Gamma^0_{ij} = a^2 f(a)\delta_{ij}$ and $\Gamma^i_{i0} = \Gamma^i_{0j} = f(a)\delta^i_j$, stating the function $f(a)$ explicitly ! *(5 points)*

c) Use the result from b) to express the equation of motion of a free 'particle' in this spacetime in terms of its 3-momentum $p = |\mathbf{p}|$. What kinds of astrophysical objects do 'particles' here (not) refer to? *(5 points)*
[Hint: try to use the normalization of the 4-momentum !]

d) Finally, show that this implies $\mathbf{p} \propto a^{-1}$, independently of the initial momentum of the particle, and interpret the result physically! *(3 points)*

Useful formulae

$$\Gamma^\mu_{\rho\sigma} = \frac{1}{2} g^{\mu\nu} \left(g_{\rho\nu,\sigma} + g_{\nu\sigma,\rho} - g_{\rho\sigma,\nu} \right) \tag{3}$$

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} \tag{4}$$

$$\delta^\mu{}_\nu = \begin{cases} 1 & \text{if } \mu = \nu \\ 0 & \text{if } \mu \neq \nu \end{cases} \tag{5}$$

$$\tilde{\epsilon}_{i_1 \ldots i_n} = \begin{cases} 1 & \text{for } i_1 \ldots i_n \text{ being an even permutation of 1,2,...,n} \\ -1 & \text{for } i_1 \ldots i_n \text{ being an odd permutation of 1,2,...,n} \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

$$\det A = \sum_{i_1 \ldots i_n} \tilde{\epsilon}_{i_1 \ldots i_n} a_{i_1 1} \ldots a_{i_n n} \,, \tag{7}$$

where the latter holds for $i_i \in \{1, ..., n\}$ and any $n \times n$ matrix A with components $\{a_{ij}\}$.