# Solutions [and grading guidelines] to final exam in FYS4160

**[General grading guidelines: Stated points are given for arriving at the respective expression in a fully satisfactory way. Whole point subtractions for bad logic, physically wrong statements or expressions that are *physically* wrong (e.g. wrong dimensions)! 'Obvious' small math errors (e.g. wrong prefactors) only 0.5 pt. When possible, no subtraction for follow-up mistakes. Up to 1 extra point for outstanding explanations or demonstrating special insight (but this cannot result in more points than available for a given problem).]**

**Problem 1**

**a)** The Einstein equivalence principle (EEP) states that in small enough regions of spacetime, the laws of physics reduce to those of special relativity. It is therefore impossible to detect the existence of a gravitational field by means of local experiments (i.e. for point-like observers, or when taking into account only infinitesimal distances). The principle implies that gravity is universal. All objects have the *same* gravitational 'charge' and there exists no gravitationally neutral object (which is the decisive difference to other forces). This implies that the notion of acceleration due to gravity is ambiguous and that we should define unaccelerated motion as motion in free fall. Since a force is something that leads to acceleration, by Newton's second law, we conclude that gravity can *not* be a force but should rather be a manifestation of a fundamental feature of spacetime, i.e. the background with respect to which motion is observed. The local features of spacetime are fully captured by its curvature. **[1 pt for principle. 1 pt for geometry connection. 1 pt for difference. ]**

**b)** The motion of a test particle in *both* special (SR) and general relativity (GR) is given by the geodesic equation:

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = f^\mu\,, \tag{1}$$

where $\tau$ is the eigentime (or an affine parameter) and $f^\mu = 0$ if there is no external force. The *difference* is that in SR the metric is always that of flat spacetime (though it might be expressed in 'strange' coordinates, such that it does not take the form of the Minkowski metric). In SR, furthermore, gravity would appear as an external force with $f^\mu \neq 0$ (though it turns out that it is not possible to consistently construct such a 4-vector), while for GR the effect of gravity is fully included in the left-hand side of the geodesic equation (by allowing for non-flat spacetimes). **[1 pt for geodesic equation for GR, and another one for SR. 1 pt for correctly pointing out the difference.]**

**c)** Since gravity is purely geometric in nature, its equations must consist of geometric invariants; in particular, all equations satisfying the strong equivalence principle must remain invariant under arbitrary changes of coordinates $x^\mu \to x'^\mu(x)$; tensors in GR are defined wrt. such general changes of coordinates **[1 pt]**. Under such a transformation, in particular, we have

$$\delta \equiv \delta^\mu{}_\nu\, \partial_\mu \otimes dx^\nu \to \delta^\mu{}_\nu \frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x^\nu}{\partial x'^\sigma}\partial'_\rho \otimes dx'^\sigma \equiv \delta'^\rho{}_\sigma \partial'_\rho \otimes dx'^\sigma\,, \tag{2}$$

which is the standard transformation law for the components of a rank $(1,1)$ tensor. The claim then follows by noting that $\delta'^{\rho}{}_{\sigma} = (\partial x'^{\rho}/\partial x^{\mu}) \times (\partial x^{\mu}/\partial x'^{\sigma}) = \partial x'^{\rho}/\partial x'^{\sigma} = \delta^{\rho}{}_{\sigma}$, where we used the chain rule in the second step **[0.5 pt for realizing what to show, 1 pt for the actual proof]**.
For the Levi-Civita symbol, on the other hand, we find **[1 pt for showing this]**

$$\tilde{\epsilon}'_{\nu\mu\sigma\rho} \equiv \tilde{\epsilon}_{\bar{\nu}\bar{\mu}\bar{\sigma}\bar{\rho}} \frac{\partial x^{\bar{\nu}}}{\partial x'^{\nu}} \frac{\partial x^{\bar{\mu}}}{\partial x'^{\mu}} \frac{\partial x^{\bar{\rho}}}{\partial x'^{\rho}} \frac{\partial x^{\bar{\sigma}}}{\partial x'^{\sigma}} = \left| \frac{\partial x}{\partial x'} \right| \tilde{\epsilon}_{\nu\mu\sigma\rho} \,, \tag{3}$$

where $|\partial x/\partial x'|$ denotes the Jacobi determinant and we used the representation of a determinant provided on the exam sheet, as well as the fact that an even (odd) permutation of columns in a matrix does not change the determinant (changes it only by an overall sign), to conclude that

$$\det A = \sum_{i_1 \ldots i_n} \tilde{\epsilon}_{i_1 \ldots i_n} a_{i_1 1} \ldots a_{i_n n} \rightsquigarrow \det A \times \tilde{\epsilon}_{j_1 \ldots j_n} = \sum_{i_1 \ldots i_n} \tilde{\epsilon}_{i_1 \ldots i_n} a_{i_1 j_1} \ldots a_{i_n j_n} \,, \tag{4}$$

for any set of $\{j_i = 1 \ldots n\}$. In other words, the transformation of $\tilde{\epsilon}_{\nu\mu\sigma\rho}$ is that of a tensor *density*, not a tensor **[0.5 pt]**.
The determinant of the metric, $g$, is also a tensor density, which allows to introduce the components of the Levi-Civita *tensor* as **[0.5 pt]**

$$\epsilon_{\nu\mu\sigma\rho} \equiv \sqrt{-g}\, \tilde{\epsilon}_{\nu\mu\sigma\rho} \,. \tag{5}$$

Being the uniquely defined $n$-form in $n$ dimensions, this tensor can be interpreted as the unit volume element for integrations over a manifold; an integral over a scalar function $\phi$ is thus given as $\int \phi \epsilon = \int d^n x \sqrt{-g} \phi(x)$ **[0.5 pt]**.

**Problem 2**

**a)** We write $R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu} = g^{\alpha\beta} R_{\alpha\mu\beta\nu}$, and therefore **[1 pt]**

$$R_{\nu\sigma} = \frac{1}{L^2} g^{\mu\rho} \left( g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho} \right) = \frac{1}{L^2} \Bigg( \underbrace{\delta_{\rho}{}^{\rho}}_{n} g_{\nu\sigma} - \underbrace{\delta_{\sigma}{}^{\rho} g_{\nu\rho}}_{g_{\nu\sigma}} \Bigg) = \frac{n-1}{L^2} g_{\nu\sigma} \tag{6}$$

Hence, $R = g^{\nu\sigma} R_{\nu\sigma} = \frac{n-1}{L^2} g^{\nu\sigma} g_{\nu\sigma} = \frac{n(n-1)}{L^2}$ **[1 pt]**.

**b)** From the solution in a), we read off **[0.5 pt each]**

$$R_{\mu\nu\rho\sigma} = \frac{R}{n(n-1)} \left( g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho} \right) \,, \tag{7}$$

$$R_{\nu\sigma} = \frac{R}{n} g_{\nu\sigma} \,. \tag{8}$$

Following the same steps as in a), it is clear that these expressions are valid for arbitrary (not only positive) values of $R$ **[1 pt]**. In other words the overall change in sign is already taken into account by the fact that $R < 1$ rather than $R > 1$.

**c)** Einstein's equations in vacuum read **[1 pt]**

$$0 = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \left(\frac{n-1}{L^2} - \frac{1}{2}\frac{n(n-1)}{L^2} + \Lambda\right) g_{\mu\nu} \tag{9}$$

Hence,

$$\Lambda = L^{-2}\left[-(n-1) + \frac{1}{2}n(n-1)\right] = \frac{1}{2}L^{-2}(n-1)(n-2)\,, \tag{10}$$

or $L = \sqrt{(n-1)(n-2)/(2\Lambda)}$ **[1 pt]**.

**d)** The line element is a that of a flat Friedman-Robertson-Walker universe, with $a(t) = \exp(Ct)$. We can thus use the first Friedmann equation to determine $C$ **[1 pt]**:

$$\left(\frac{\dot{a}}{a}\right)^2 = C^2 = \frac{8\pi G}{3}\rho_\Lambda = \frac{\Lambda}{3} \quad \rightsquigarrow \quad C = \sqrt{\Lambda/3}\,. \tag{11}$$

The 4D-curvature of de-Sitter space is indeed positive (as derived in problem c); however, for the specific foliation of this spacetime into a time-like direction and a spherically symmetric submanifold in 3D the *induced metric* on the latter describes a space without (3D) curvature **[1 pt]**.

**Problem 3**

**a)** In inertial coordinates, the 4-acceleration is given by $du^\mu/d\tau$, where $u^\mu = dx^\mu/d\tau$ is the four-velocity. In non-inertial coordinates, or for accelerated observers, this generalizes to $a^\mu = Du^\mu/d\tau = u^\nu\partial_\nu u^\mu + \Gamma^\mu_{\rho\sigma}u^\rho u^\sigma$ **[1 pt]**. For a stationary observer we have $u^i = 0$, and hence **[1 pt]**

$$a^\mu = (u^0)^2\Gamma^\mu_{00} = \left(1 - \frac{r_s}{r}\right)^{-1}\Gamma^\mu_{00}\,, \tag{12}$$

where the second equality follows from the $-1 = u_\mu u^\mu = g_{00}(u^0)^2$. Using that the relevant metric components only have radial dependence, in spherical coordinates, we can calculate the above Christoffel symbol as $\Gamma^\mu_{00} = \frac{1}{2}g^{\mu\nu}\left(g_{0\nu,0} + g_{\nu 0,0} - g_{00,\nu}\right) = -\frac{1}{2}g^{\mu r}g_{00,r}$, implying that only the radial component ($\mu = r$) is non-vanishing, with **[1 pt]**

$$\Gamma^r_{00} = \frac{1}{2}\left(1 - \frac{r_s}{r}\right)\partial_r\left(1 - \frac{r_s}{r}\right) = \frac{r_s}{2r^2}\left(1 - \frac{r_s}{r}\right)\,. \tag{13}$$

The proper acceleration, i.e. the magnitude of the 3-acceleration that needs to be balanced by the spacecraft, is therefore **[1 pt]**

$$a \equiv \sqrt{g_{ij}a^i a^j} = \sqrt{g_{rr}}a^r = \frac{r_s}{2r^2}\left(1 - \frac{r_s}{r}\right)^{-\frac{1}{2}}\,. \tag{14}$$

The first factor, $r_s/2r^2 = GM/r^2$, is the Newtonian result, while the factor $\left(1 - \frac{r_s}{r}\right)^{-1/2} = \sqrt{1 + r_s/R}$ is the relativistic correction. It diverges as $R \to 0$: staying 'at rest' closer and closer to the horizon requires an increasingly larger acceleration **[1 pt]**.

**b)** We first recall that $\partial_t$ is a Killing vector in this spacetime, implying that

$$E \equiv -g_{\mu\nu}(\partial_t)^\mu u^\nu = -g_{00}u^0 = \left(1 - \frac{r_s}{r}\right)\frac{dt}{d\tau} \tag{15}$$

is conserved on geodesics, i.e. during free fall **[1 pt]**. At the onset of free fall, for $r = r_s + R$, we have $u^i = 0$; making use of the normalization condition $-1 = u_\mu u^\mu = g_{00}(u^0)^2$ therefore gives **[1 pt]**

$$E = \left(1 - \frac{r_s}{r}\right)\sqrt{-1/g_{00}}\Big|_{r=r_s+R} = \left(\frac{R}{r_s + R}\right)^{\frac{1}{2}} . \tag{16}$$

In our normalization, the conserved energy thus varies between $E = 0$ (when releasing very close to the even horizon) and $E = 1$ (corresponding to the maximal potential energy in the system, for a release at $R \to \infty$); for $r < r_s + R$, furthermore, we always have $1 - E^2 < r_s/r$.
We can now consider time-like geodesics **[1 pt]**,

$$d\tau^2 = \left(1 - \frac{r_s}{r}\right)dt^2 - \left(1 - \frac{r_s}{r}\right)^{-1}dr^2 = \left(1 - \frac{r_s}{r}\right)^{-1}\left(E^2 d\tau^2 - dr^2\right), \tag{17}$$

i.e. $(dr/d\tau)^2 = r_s/r + E^2 - 1$, and integrate from $r = r_s + R$ in order to obtain the eigentime until the spacecraft reaches the singularity at $r = 0$ **[2 pts for final result]**:[1]

$$\tau = \int_{r_s+R}^{0} \frac{dr}{\sqrt{E^2 - 1 + r_s/r}} = 2r_s(1 - E^2)^{-3/2}\int_0^\infty \frac{dx}{(1 + x^2)^2} = \frac{\pi}{2}r_s\left(1 + R/r_s\right)^{\frac{3}{2}} , \tag{18}$$

where we performed a change of variables to $\sqrt{1 - E^2}x = \sqrt{-1 + E^2 + r_s/r}$. If the spacecraft is released from a static position very close to the horizon ($R \to 0$), we recover the expression $\tau = \pi GM$ previously derived in one of the exercises; if the free fall starts far away from the black hole ($R \gg r_s$), it takes $\tau = 2^{-3/2}\pi(GM)^{-1/2}R^{3/2}$ to reach the singularity. For the Newtonian case, on the other hand, an acceleration of $\ddot{r} = GM/r^2$, implies that it takes $t = 2^{1/2}(GM)^{-1/2}R^2$ to reach the center. The mismatch in the scaling with $R$ is a result of time dilation, which has no Newtonian counterpart **[1 pt]**.

**c)** Even for non-inertial trajectories, the four-velocity is normalized as

$$-1 = g_{\mu\nu}u^\mu u^\nu = -\left(1 - \frac{r_s}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + \left(1 - \frac{r_s}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 \tag{19}$$

$$= \left|1 - \frac{r_s}{r}\right|\left(\frac{dt}{d\tau}\right)^2 - \left|1 - \frac{r_s}{r}\right|^{-1}\left(\frac{dr}{d\tau}\right)^2 , \tag{20}$$

[1] There is a subtlety here in that the Schwarzschild coordinates only cover the region $r > r_s$ of the manifold. However, they can also be used to describe the region $r < r_s$, even though $r$ then describes the *time* coordinate and $t$ the radial, spatial coordinate. Since $\partial_t$ is a Killing vector in both regions, however, the calculation still works in exactly the same way in both regions. Excluding the point $r = r_s$ and/or just adding the contribution from $r > r_s$ and $r < r_s$, respectively, does not change the result of the integral in Eq. (18) **[up to 2 bonus points for noting and discussing this]**.

where we assumed, as indicated, radial motion, and the second step is valid inside the horizon **[1 pt]**. Since the first term is strictly positive and the second strictly negative, the equation can only be satisfied for **[1 pt]**

$$\left|1-\frac{r_s}{r}\right|^{-1}\left(\frac{dr}{d\tau}\right)^2 \geq 1 \quad \rightsquigarrow \quad \frac{dr}{d\tau} \leq -\sqrt{r_s/r-1}\,, \tag{21}$$

where only the negative solution is possible because $r$ must be decreasing (recalling that $r$ is time-like inside the horizon, this corresponds to increasing time) **[0.5 pt]**. Maximizing the time spent in the black hole is achieved by satisfying the equality in the above expression; comparing to the result from b), this corresponds to free fall with $E = 0$, i.e. a negligible kinetic energy at the moment of passing through the horizon. Thus, both a larger value of $E$ while freely falling and a non-inertial motion can only shorten the time it takes to reach the singularity. The recommendation is thus clear: the crew should use maximal outward-directed thrust for $r > r_s$ (in order to minimize $E$), and immediately switch off the engine once they have passed the horizon (because non-inertial motion will only shorten their lifetime) **[1 pt]**. From Eq. (18), the maximal time thus spent inside the black hole is

$$\tau = \frac{\pi}{2} r_s \approx \frac{\pi}{2} \frac{2.9\,\text{km}}{c = 3\cdot 10^5\text{km/s}} \frac{M}{M_\odot} \approx 3000\,\text{s}\,, \tag{22}$$

i.e. about 50 minutes **[0.5 pt]**.

**<u>Problem 4</u>**

**a)** Vanishing spatial curvature corresponds to $\kappa = 0$ **[0.5 pt]**. Expressing the line element in Cartesian coordinates, rather than spherical (or polar) coordinates **[0.5 pt]**, then gives **[1 pt]**

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j\,. \tag{23}$$

The effect of curvature only enters in the 1st Friedmann equation, according to which the total energy density is proportional to $(1+\kappa/\dot{a}^2)$; since $\dot{a}$ is monotonically decreasing with time, a non-vanishing curvature ($\kappa \neq 0$) thus necessarily had a smaller effect on the evolution of the early universe than it has today **[1 pt]**.

**b)** From the form of the line element in Eq. (23) we see that $\partial_\rho g_{\mu\nu} = 0$ for $\rho \neq 0$ **[1 pt]**. Thus **[1 pt for each of these results]**,

$$\Gamma^0_{\rho\sigma} = \frac{1}{2}g^{0\nu}\left(g_{\rho\nu,\sigma}+g_{\nu\sigma,\rho}-g_{\rho\sigma,\nu}\right) = -\frac{1}{2}\left(g_{\rho 0,\sigma}+g_{0\sigma,\rho}-\partial_t g_{\rho\sigma}\right) = \frac{1}{2}\partial_t g_{\rho\sigma}\,, \tag{24}$$

$$\Gamma^i_{\rho\sigma} = \frac{1}{2}g^{i\nu}\left(g_{\rho\nu,\sigma}+g_{\nu\sigma,\rho}-g_{\rho\sigma,\nu}\right) = \frac{1}{2}a^{-2}\left(g_{\rho i,\sigma}+g_{i\sigma,\rho}\right)\,, \tag{25}$$

where we further used $\partial_\rho g_{0\nu} = 0$ and that the metric is diagonal ($g_{\mu\nu} = 0$ for $\mu \neq \nu$). The first term is only non-vanishing for $\rho = \sigma = i$, giving $\Gamma^0_{ij} = a^2 H \delta_{ij}$, where $H = \dot{a}/a$ **[1 pt]**. For the second term non-vanishing contributions can only appear when one of the indices $\rho, \sigma$ equals $i$ (because the metric is diagonal) and the other one equals 0 (because all other derivatives give zero); this gives $\Gamma^i_{j0} = \Gamma^i_{0j} = H\delta^i_j$, i.e. $f(a) = H$ **[1 pt]**.

**c)** A free particle follows the geodesic equation, $\frac{d}{d\tau}u^\mu = -\Gamma^\mu_{\rho\sigma}u^\rho u^\sigma$ with $u^\mu = dx^\mu/d\tau$, and hence **[1 pt]**

$$\frac{du^0}{d\tau} = -a^2 H\delta_{ij}u^i u^j = -H\mathbf{u}^2\,, \tag{26}$$

where $\mathbf{u}^2 = g_{ij}u^i u^j$. Following the hint, we recall that time-like four-velocities are normalized as $g_{\mu\nu}u^\mu u^\nu = -1$ **[1 pt]**. Therefore,

$$\frac{d}{d\tau}\left(u^0\right)^2 = \frac{d}{d\tau}\left(1+\mathbf{u}^2\right) \quad \rightsquigarrow \quad u^0\frac{du^0}{d\tau} = |\mathbf{u}|\,\frac{d\,|\mathbf{u}|}{d\tau}\,.\text{[1 pt]} \tag{27}$$

Noting that $dt = u^0 d\tau$ and $\mathbf{p} = m\mathbf{u}$, this allows us to rewrite Eq. (26) as **[1 pt]**

$$\frac{d\,|\mathbf{p}|}{dt} = -H\,|\mathbf{p}|\;. \tag{28}$$

This equation holds for non-interacting objects that feel no other effect of gravity than that of the expanding universe, as described by the FRW spacetime; the smallest such objects are entire galaxies – while planets, stars, etc are parts of gravitationally bound systems and hence do not experience the effect of the expansion **[1 pt]**.

**d)** We first note that the last equation directly implies $d\mathbf{p}/dt = -H\mathbf{p}$, because of the spherical symmetry of the space-time **[1 pt]**. We can solve this differential equation by integrating by parts **[1 pt]**:

$$\frac{d(\log p^i)}{dt} = -a^{-1}\frac{da}{dt} = -\frac{d(\log a)}{dt} \quad \rightsquigarrow \quad p^i \propto a^{-1}\,. \tag{29}$$

The physical interpretation is that, for non-interacting particles, each component of the 3-momentum simply redshifts with the expansion of the universe **[1 pt]**.

**[ The final grade is calculated from the total number of points $P$ as follows:**

$P \geq 11$ **: E (pass)**
$P \geq 17$ **: D**
$P \geq 23$ **: C**
$P \geq 29$ **: B**
$P \geq 35$ **: A ]**