Motion in non-inertial coordinates / in gravitational field

in free-fall coordinates $\xrightarrow[\text{EP}]{}$ straight line, i.e.

$$\boxed{\frac{d^2\xi^\mu}{d\tau^2} = 0}$$

now transform to $x^\mu = x^\mu(\xi)$

$$0 = \frac{d^2\xi^\mu}{d\tau^2} = \frac{d}{d\tau}\Big(\underbrace{\frac{\partial\xi^\mu}{\partial x^\nu}\frac{dx^\nu}{d\tau}}_{\frac{d\xi^\mu}{d\tau}}\Big)$$

(chain rule!)

$$= \frac{\partial\xi^\mu}{\partial x^\nu}\frac{d^2x^\nu}{d\tau^2} + \frac{dx^\nu}{d\tau}\Big(\underbrace{\frac{\partial^2\xi^\mu}{\partial x^\nu\partial x^\rho}\frac{dx^\rho}{d\tau}}_{\frac{d}{d\tau}\left(\frac{\partial\xi^\mu}{\partial x^\nu}\right)}\Big) \qquad \Big|\times\frac{\partial x^\sigma}{\partial\xi^\mu}$$

$$\Rightarrow 0 = \underbrace{\frac{\partial x^\sigma}{\partial\xi^\mu}\frac{\partial\xi^\mu}{\partial x^\nu}}_{\frac{\partial x^\sigma}{\partial x^\nu}=\delta^\sigma_\nu}\frac{d^2x^\nu}{d\tau^2} + \underbrace{\frac{\partial x^\sigma}{\partial\xi^\mu}\frac{\partial^2\xi^\mu}{\partial x^\nu\partial x^\rho}}_{\equiv\Gamma^\sigma_{\nu\rho}}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}$$

(chain rule!)

$\equiv \Gamma^\mu_{\nu\rho}$ "Christoffel symbols"

$$\Rightarrow \boxed{\frac{d^2x^\sigma}{d\tau^2} + \Gamma^\sigma_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0}$$

– geodesic equation
= eq. of motion in arbitrary coordinates

and / or spacetimes!

introduce inverse metric:

$g^{\rho\sigma}$, i.e. $g^{\mu\rho} g_{\rho\sigma} = \delta^{\mu}_{\sigma}$

(→ exercises...)

$$\boxed{\Gamma^{\sigma}_{\mu\nu} = \frac{g^{\sigma\rho}}{2}\left(g_{\nu\rho,\mu} + g_{\mu\rho,\nu} - g_{\mu\nu,\rho}\right)}$$

What is the geometric interpretation of geodesics?

→ extremal distance between any two points A, B:

$$\int_A^B ds = \text{extremal} \Leftrightarrow \delta \int_A^B ds \overset{!}{=} 0$$

NB: coordinate-invariant

choose parameter λ

⇝ consider time-like curves $x^{\mu}(\lambda)$

$$\Rightarrow -ds^2 = +d\tau^2 = -g_{\mu\nu} \frac{dx^{\mu}}{d\lambda} \frac{dx^{\nu}}{d\lambda} d\lambda^2 \equiv -g_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu} d\lambda^2 > 0$$

$$\Rightarrow \quad \delta \int \underbrace{\sqrt{-g_{\mu\nu}^{(x)} \dot{x}^\mu \dot{x}^\nu}}_{\equiv L(x,\dot{x})} d\lambda = 0$$

$\Rightarrow$ Lagrange eq. of motion:

$$\boxed{\frac{\partial L}{\partial x^\mu} = \frac{d}{d\lambda} \frac{\partial L}{\partial \dot{x}^\mu}} \quad (*)$$

- $$\frac{\partial L}{\partial x^\mu} = -\frac{1}{2L} g_{\rho\sigma,\mu} \dot{x}^\rho \dot{x}^\sigma$$

- $$\frac{\partial L}{\partial \dot{x}^\mu} = \frac{1}{2L} (-g_{\rho\sigma}) \underbrace{\frac{\partial}{\partial \dot{x}^\mu} (\dot{x}^\rho \dot{x}^\sigma)}_{\underbrace{\frac{\partial \dot{x}^\rho}{\partial \dot{x}^\mu}}_{\delta^\rho_\mu} \dot{x}^\sigma + \dot{x}^\rho \underbrace{\frac{\partial \dot{x}^\sigma}{\partial \dot{x}^\mu}}_{\delta^\sigma_\mu}}$$

$$= -\frac{1}{2L}\left(g_{\mu\sigma} \dot{x}^\sigma + g_{\rho\mu} \dot{x}^\rho \right) = -\frac{1}{L} \dot{x}_\mu \ (x,\dot{x})$$

$$\Rightarrow \frac{d}{d\lambda} \frac{\partial L}{\partial \dot{x}^\mu} = -\frac{d}{d\lambda}\left(\frac{1}{L} g_{\mu\nu} \dot{x}^\nu \right)$$

$$= \frac{\dot{L}}{L^2} g_{\mu\nu} \dot{x}^\nu - \frac{1}{L} \underbrace{\dot{g}_{\mu\nu}}_{= \frac{d}{d\lambda} g_{\mu\nu} = \frac{\partial g_{\mu\nu}}{\partial x^\rho} \frac{dx^\rho}{d\lambda} = g_{\mu\nu,\rho} \dot{x}^\rho} \dot{x}^\nu - \frac{1}{L} g_{\mu\nu} \ddot{x}^\nu$$

(#)

$$\Rightarrow -\frac{1}{2L} g_{\rho\sigma,\mu}\,\dot{x}^\rho \dot{x}^\sigma = \frac{\dot{L}}{L^2} g_{\mu\nu}\dot{x}^\nu - \frac{1}{L} g_{\mu\nu,\rho}\dot{x}^\rho \dot{x}^\nu - \frac{1}{L} g_{\mu\nu}\ddot{x}^\nu \qquad \Big|\times g^{\tau\mu}$$

$$\Leftrightarrow \underbrace{\delta^\tau_\nu \ddot{x}^\nu}_{\ddot{x}^\tau} + \underbrace{\frac{1}{2} g^{\tau\mu}\big(\underbrace{2 g_{\mu\sigma,\rho}}_{g_{\mu\sigma,\rho}+g_{\mu\rho,\sigma}} - g_{\rho\sigma,\mu}\big)\dot{x}^\rho \dot{x}^\sigma}_{=\Gamma^\tau_{\sigma\rho}\dot{x}^\rho \dot{x}^\sigma} = \frac{\dot{L}}{L^2}\underbrace{\delta^\tau_\nu \dot{x}^\nu}_{\dot{x}^\tau}$$

[NB: not "equal to"!]

Now choose $\lambda = \tau$ (#) [also works "affine parameter" $\lambda = a + b\tau$]

$$\Rightarrow L^2 = -g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = +1 = \text{const.}$$

$$\Rightarrow \dot{L} = \frac{dL}{d\tau} = 0$$

$$\Rightarrow \boxed{\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = 0}$$

$\Rightarrow$ geodesics = "straight lines in arbitrary coordinates or spacetimes"

NB: (*) does not work for photons!

BUT: geodesic eq. still takes the same form, with τ replaced by arbitrary parameter λ ($\rightsquigarrow$ exercises)

$$\boxed{\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau} = 0}$$

recall Newton: $\frac{d\vec{x}^2}{dt^2} - \frac{\vec{F}}{m} = 0 \iff \frac{d^2x^\mu}{d\tau^2} - \frac{f^\mu}{m} = 0$

$\hookrightarrow$ $\Gamma^\mu_{\rho\sigma}$ purely geometric objects

(i.e. they only depend on metric)

$\Rightarrow$ as anticipated, gravity can be described without ordinary force!

instead ("essence of GR")

- distribution of mass/energy curves space-time — TBD
- particles perform a force-free (free fall) motion in this spacetime — ✓

$\Rightarrow$ outlook: I) develope (more) mathematical tools to describe curved space(-time)

"differential geometry"

II) construct field equations (for $g_{\mu\nu}$)

"geometry = matter (and energy) distribution"

2.3 Examples (that do <u>not</u> require I, II)

A) Newtonian limit

($\equiv$ static, small gravitational field)

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x) \; ; \; |h_{\mu\nu}| \ll 1$$

["there is a coordinate system such that..."]

$\Rightarrow$ geodesic equation:

$$\frac{d^2x^\mu}{d\tau^2} = -\Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau}$$

$\Big|\; v \ll 1 \Rightarrow \frac{dx^\mu}{d\tau} = u^\mu = \gamma(1, \vec{v}) \simeq (1, \vec{0})$

$$\simeq -\Gamma^\mu_{00}\left(\frac{dt}{d\tau}\right)^2$$

$\Big|\; \Gamma^\mu_{00} = \frac{1}{2} g^{\mu\nu}\Big(\underbrace{\cancel{g_{\nu 0,0}} + \cancel{g_{0\nu,0}}}_{=0 \text{ (static!)}} - \underbrace{g_{00,\nu}}_{= h_{00,\nu}}\Big)$

$$\simeq -\frac{1}{2}\eta^{\mu\nu} h_{00,\nu}$$

(only keep first order terms)

$\Rightarrow \mu=0)\quad \eta^{\overset{0}{\cancel{\mu}}\nu} h_{00,\nu} = \eta^{00} h_{00,0} = 0$ (static!)

$$\Rightarrow \frac{d^2t}{d\tau^2} = 0 \quad \Rightarrow \quad \frac{dt}{d\tau} = \text{const} \longrightarrow 1 \text{ after rescaling}$$

(coordinate trafo $t \Rightarrow t \times \text{const.}$)

$$\mu=i)\quad \frac{d^2x^i}{d\tau^2} \propto \frac{d^2x^i}{dt^2} = +\frac{1}{2}\eta^{i\nu} h_{00,\nu}$$

$$= \frac{1}{2}\eta^{ij} h_{00,j}$$

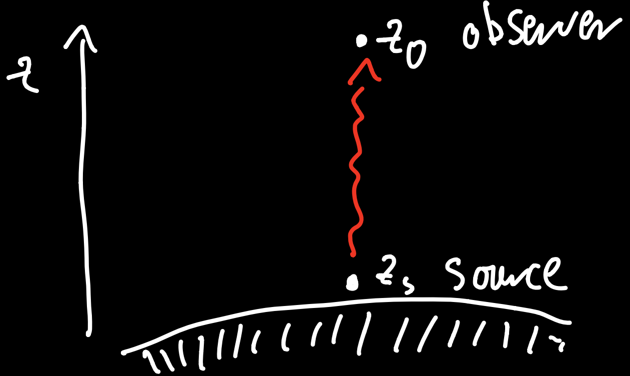
$$= \frac{1}{2}(\nabla h_{00})^i$$

$$\overset{!}{=} -(\vec{\nabla}\phi_N)^i \quad \text{[Newtonian gravity!]}$$

$$\Rightarrow \boxed{g_{00} = -(1+2\phi_N)}$$

⌜NB: no integration constant because of EP!!! ⌟

B) Gravitational red-shift



$$d\tau^2 = -g_{\mu\nu}dx^\mu dx^\nu$$

$$= -(1+2\phi)dt^2 - dx^2 - dy^2 - dz^2$$

$$@ \text{ rest}: \quad d\tau_{0,s} = \sqrt{1+2\phi_{0,s}}\, dt$$

$$\Rightarrow \text{frequency shift}: \quad \frac{\nu_s}{\nu_0} = \frac{d\tau_0}{d\tau_s} \overset{(*)}{=} \frac{\sqrt{1+2\phi_0}}{\sqrt{1+2\phi_s}} = 1 + 2\phi_0 - 2\phi_s + \mathcal{O}(\phi^2)$$