1. Special relativity

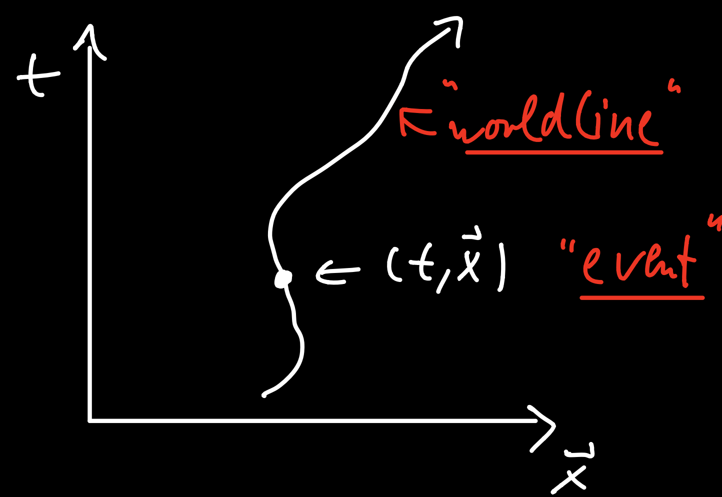
1.1 structure of spacetime

→ kinematics (⟺ dynamics: eq. of motion > forces)

postulate	Newton	Einstein
P1: "space & time are homogeneous and isotropic"	✓	✓
P2: "The laws of physics take the same form in all inertial frames [+ all inertial frames are in a state of constant, rectilinear motion w.r.t. each other]	✓	✓
PN "space & time are absolute, i.e. independent of inertial frames"	✓	✗
PE "The velocity c [of light in vacuum] is a universal constant, which takes the same value in all inertial frames"	✗	✓

NB: • P1 + P2 + PN *defines* Newton's view!
" " + PE " Einstein's " !

• "spacetime" is a common concept!
$= \{(t, \vec{x})\}$

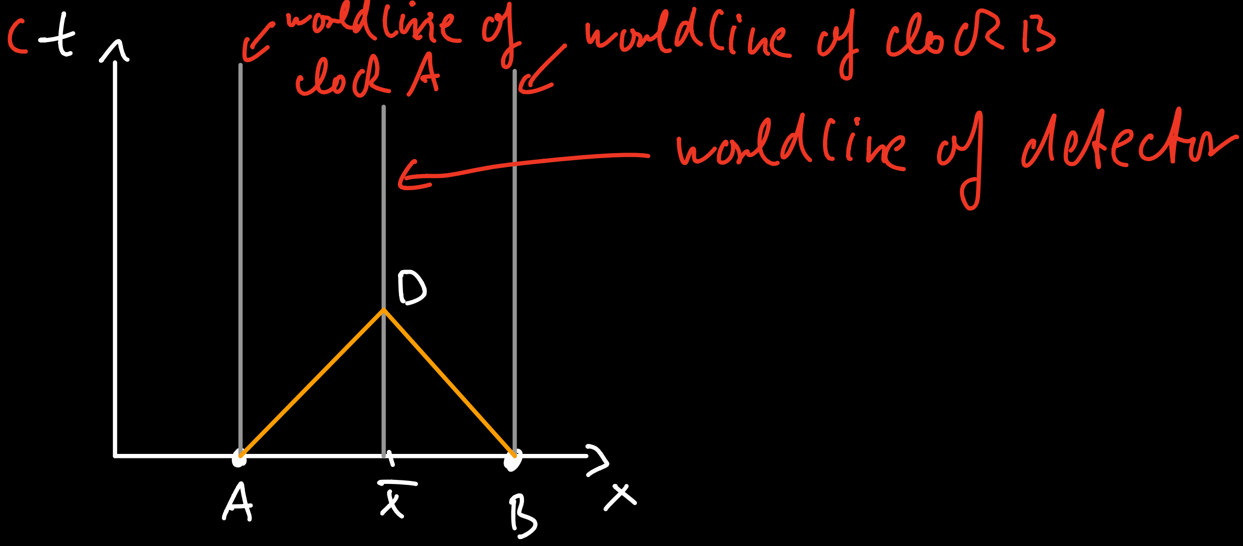


synchronization of clocks & simultaneity of events

Imagine two clocks at position x_A, x_B

Q: How to test that they are synchronized?

A: Flash a light signal at x_A & x_B when both clocks show the same time, $t_A = t_B = t_0$.
$\Rightarrow$ iff the two signals reach $\bar{x} \equiv \frac{1}{2}(x_A + x_B)$ at the same time, the clocks are *synchronized* and the two events $A = (t_0, x_A)$ and $B = (t_0, x_B)$ are *simultaneous*.



in stationary frame :

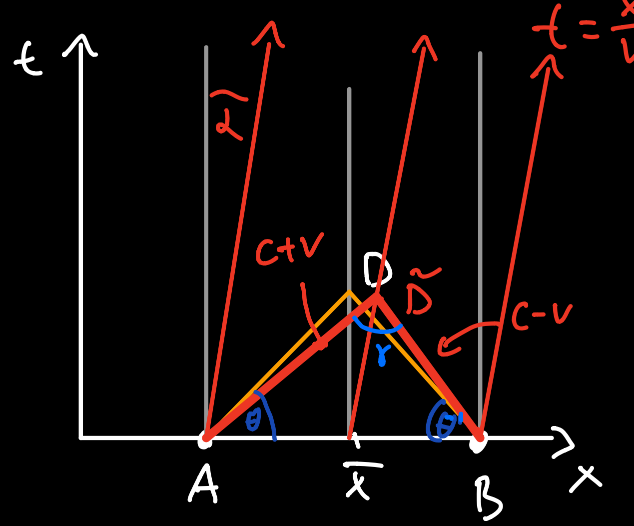
> The $\underline{x\text{-axis}}$ is given by all events simultaneous with the origin
> The $\underline{t\text{-axis}}$ is given by the worldline of a stationary observer

Now consider another inertial frame, moving with constant velocity v

Newtonian analysis

t' : worldline of moving clock A:

$t = \frac{x}{v} \Rightarrow \tan\alpha = \frac{x}{t} = v$



$\theta + \theta' = \left(\frac{\pi}{4} - \alpha\right) + \left(\frac{\pi}{4} + \alpha\right) = \frac{\pi}{2}$

$\Rightarrow \gamma = \frac{\pi}{2}$

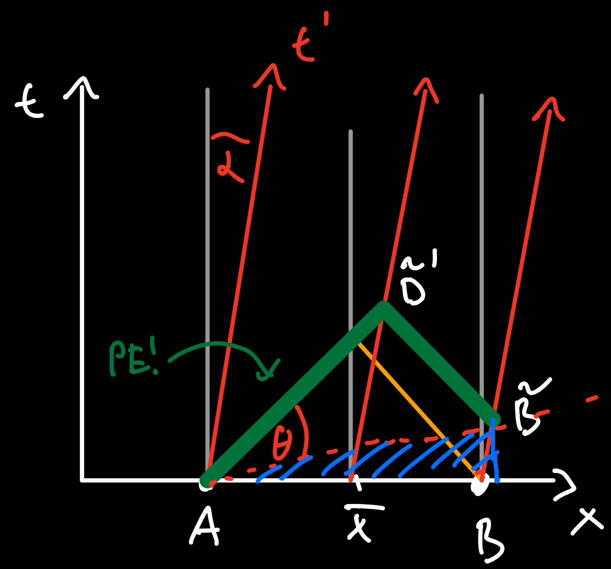
$\Rightarrow \tilde{D}$ lies on circle !

$\Rightarrow$ i) A & B are simultaneous also in all other frames (($\Leftrightarrow \tilde{B}$ always on world line of $\bar{x}$)

ii) spatial and temporal distances remain unchanged

$\Rightarrow$ $\boxed{\begin{array}{l}\Delta t = \text{const.} = \Delta t' \\ |\Delta \vec{x}| = \text{const} = |\Delta \vec{x}'|\end{array}}$

SR analysis



$\rightarrow x'$: obtained by reflecting about diagonal

$\Rightarrow \theta = \frac{\pi}{4} - \alpha$

NB: $\theta' = \frac{\pi}{4}$!

(because of PE!)

$\Rightarrow$ • A & $\tilde{B}$ (not B!) are simultaneous!

$\Leftrightarrow$ lie on the same x' axis

• $t'_B < t_B = t_A \equiv t'_A \quad \forall \underbrace{0 < v < c}$

all those observers agree that B happens first

[$v > c$ would allow the opposite $\rightsquigarrow$ causality violation]

$\Rightarrow |\Delta t|$ & $|\vec{x}|$ are not the same in all inertial frames!

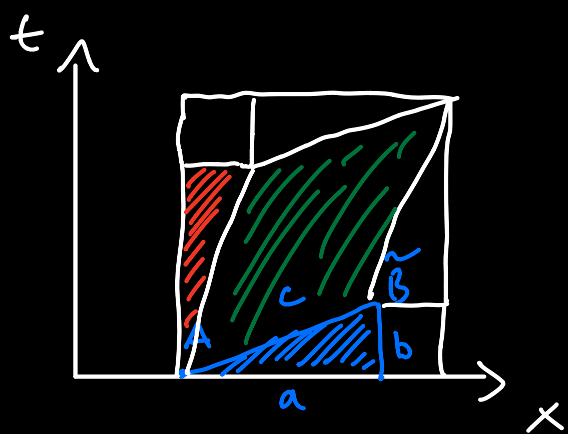
BUT the spacetime-interval is constant:

$$\boxed{(\Delta s)^2 = -(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}$$

$\rightsquigarrow$ why? a) for photons: $c = \frac{\Delta x}{\Delta t} = \frac{\Delta x'}{\Delta t'}$

$$\Rightarrow (\Delta \vec{x})^2 - (c\Delta t)^2 = 0 = (\Delta \vec{x}')^2 - (c\Delta t')^2$$

b) for all other inertial frames (moving w/ $v < c$):



$a = \Delta x$ $\quad$ Δt

$b = \Delta t$ $\quad$ Δx

$c = \Delta x'$ $\quad$ $\Delta t'$

now calculate area :

$$(a+b)^2 = 4 \cdot \frac{a \cdot b}{2} + 2b^2 + \underbrace{A}_{=c^2}$$

NB: in primed system, lines labelled w/ "c" are perpendicular!

$$a^2 + 2ab + b^2 = 2ab + 2b^2 + c^2$$

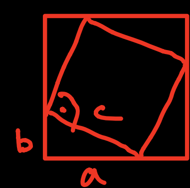
$$\Rightarrow \underbrace{c^2}_{(\Delta s')^2} = \underbrace{a^2 - b^2}_{(\Delta s)^2}$$

$$-(\Delta s')^2 \qquad -(\Delta s)^2$$

$\square$

" Pythagoras in Minkowski space "

⌜ in Euclidian space:



⌟

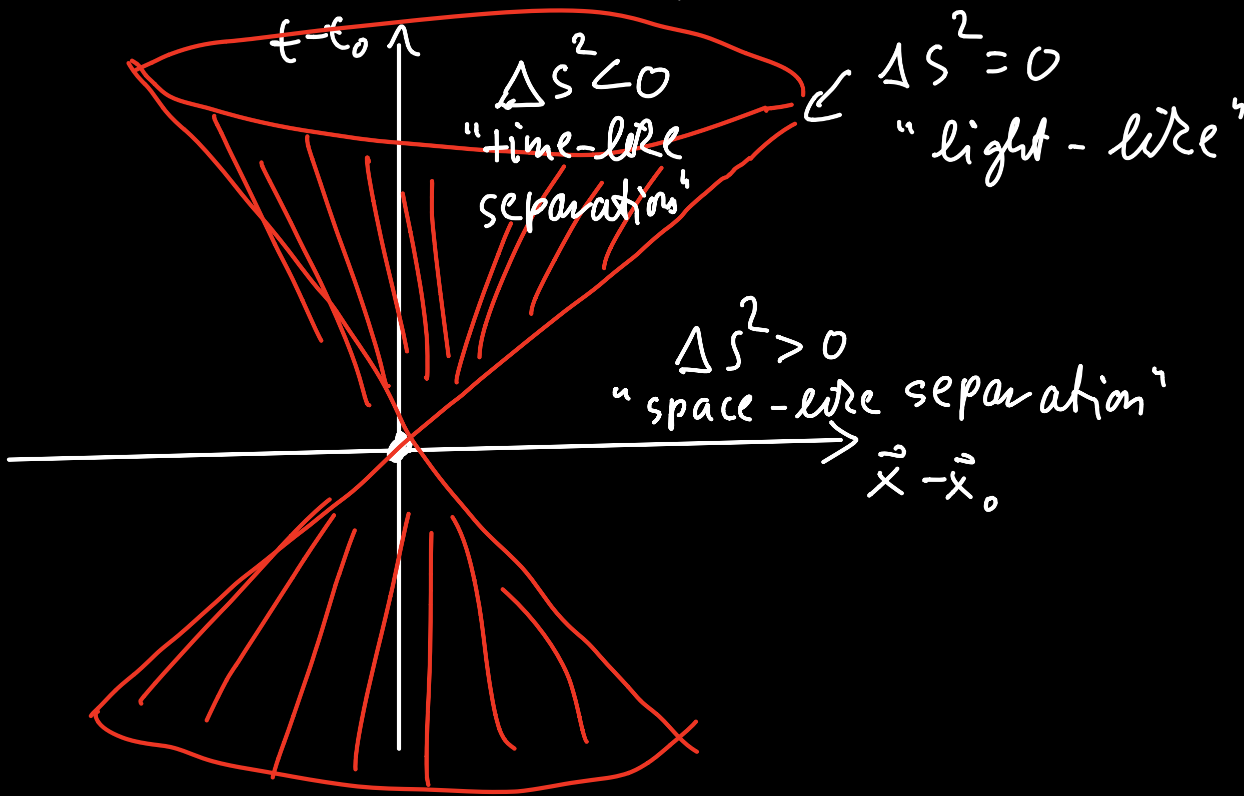
Def. <u>proper time</u> τ : $(\Delta \tau)^2 \stackrel{\text{def}}{=} -(\Delta s)^2$

= time elapsed for an observer moving on a straight between two events

⌜ observer @ rest : $(\Delta \tau)^2 = -(\Delta s)^2 = (\Delta t)^2$

$\Rightarrow \Delta \tau = \Delta t$ ⌟

@ every point $(t_0, \vec{x}_0)$ of a worldline, one can draw a spacetime diagram:



⇝ inside the <u>light cone</u>, all events are "causally connected" to observer

⌜observers in all inertial frames agree that these events happen in the future of $(t_0, \vec{x}_0)$

⇝ could in principle be causally affected

(for future light cone, $t > t_0$)

[correspondingly, everything in past light cone lies in the past and could potentially have influenced $(t_0, \vec{x}_0)$]
⌟

1.2. space-time coordinates and Lorentz transformations

$$x^\mu = \begin{cases} ct & \text{for } \mu = 0 \\ x^i & \text{for } \mu = i = 1,2,3 \end{cases}$$

introduce Minkowski metric $\eta_{\mu\nu} \equiv \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$$\Rightarrow (\Delta s)^2 = \underset{(\sum_{\mu,\nu})}{\eta_{\mu\nu}} \Delta x^\mu \Delta x^\nu \equiv \Delta x \cdot \Delta y$$

NB: sum convention!

infinitesimal version: "<u>line element</u>"

$$\boxed{\frac{ds^2}{d\lambda^2} = \eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}$$

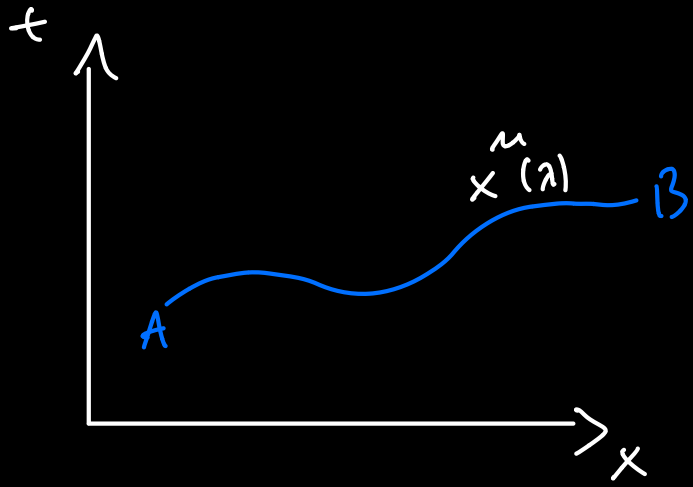
consider parameterized curve through spacetime: $x^\mu(\lambda)$

$\Rightarrow$ "length" of curve (in 4D):

i) $\Delta s = \int_A^B \sqrt{\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}\, d\lambda$ for space-like path

$\Rightarrow \eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} > 0$

ii) $\Delta \tau = \int_A^B \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}\, d\lambda$ for time-like path



$\Rightarrow$ time elapsed for observer travelling from A to B along $x^\mu(\lambda)$

NB: These expressions are perfectly valid even for accelerated trajectories!