b) always for geodesics:

$$-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} \equiv \varepsilon = \text{const.}$$

- $\lambda = \tau$ (massive particles) $\Rightarrow \varepsilon = -g_{\mu\nu}u^\mu u^\nu = +1$
- (massless particles) $\Rightarrow \varepsilon = 0$

⌈λ is instead fixed by $p^\mu = \frac{dx^\mu}{d\lambda}$⌋

$$\Rightarrow \varepsilon = \left(1-\frac{2GM}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 - \left(1-\frac{2GM}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 - r^2\left(\frac{d\Omega}{d\lambda}\right)^2$$

$= \left(\frac{d\phi}{d\lambda}\right)^2$ ↗ $\theta = \frac{\pi}{2} = \text{const.}$

- $\frac{dt}{d\lambda} = \frac{E}{\left(1-\frac{2GM}{r}\right)}$
- $\frac{d\phi}{d\lambda} = \frac{L}{r^2}$

$$\Rightarrow \varepsilon = \left(1-\frac{2GM}{r}\right)^{-1}\left\{E^2 - \left(\frac{dr}{d\lambda}\right)^2\right\} - \frac{L^2}{r^2} \quad \Big| \times -\frac{1}{2}\left(1-\frac{2GM}{r}\right)$$

$$\Leftrightarrow \boxed{\frac{1}{2}(E^2-\varepsilon) = +\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 - \varepsilon\frac{GM}{r} + \frac{L^2}{2r^2} - \frac{GML^2}{r^3}}$$

$\frac{1}{2}(E^2-\varepsilon)$ = const.

$-\varepsilon\frac{GM}{r} + \frac{L^2}{2r^2} - \frac{GML^2}{r^3}$: $U_{eff}(r)$

$\frac{GML^2}{r^3}$: not present in Newtonian

$\Rightarrow$ identical behaviour for $r \gg 2GM$!

comments:
- the above is an exact equation (not an expansion in $1/r$)
- for $m>0 \rightsquigarrow \varepsilon = +1$:

$$E = \left(1 - \frac{2GM}{r}\right)\frac{dt}{d\tau} \overset{r \gg 2GM}{\approx} 1 + \frac{E_{\text{kin}}}{m}$$

$\frac{dt}{d\tau} = \frac{m + E_{\text{kin}}}{m}$; $\frac{E_{\text{kin}}}{m} \ll 1$ in Newtonian limit

$$\Rightarrow \mathcal{E}_1 = \frac{1}{2}(E^2 - 1) = \frac{1}{2}\left(2\frac{E_{\text{kin}}}{m} + \frac{E_{\text{kin}}^2}{m^2}\right) \simeq \frac{E_{\text{kin}}}{m}$$

$\rightsquigarrow$ same as l.h.s. in Newtonian expression

$\Rightarrow \mathcal{E}_1$ is ~the kinetic energy for $r \gg 2GM$

- at smaller r: interpretation of "energy" changes, but equation still of the same form

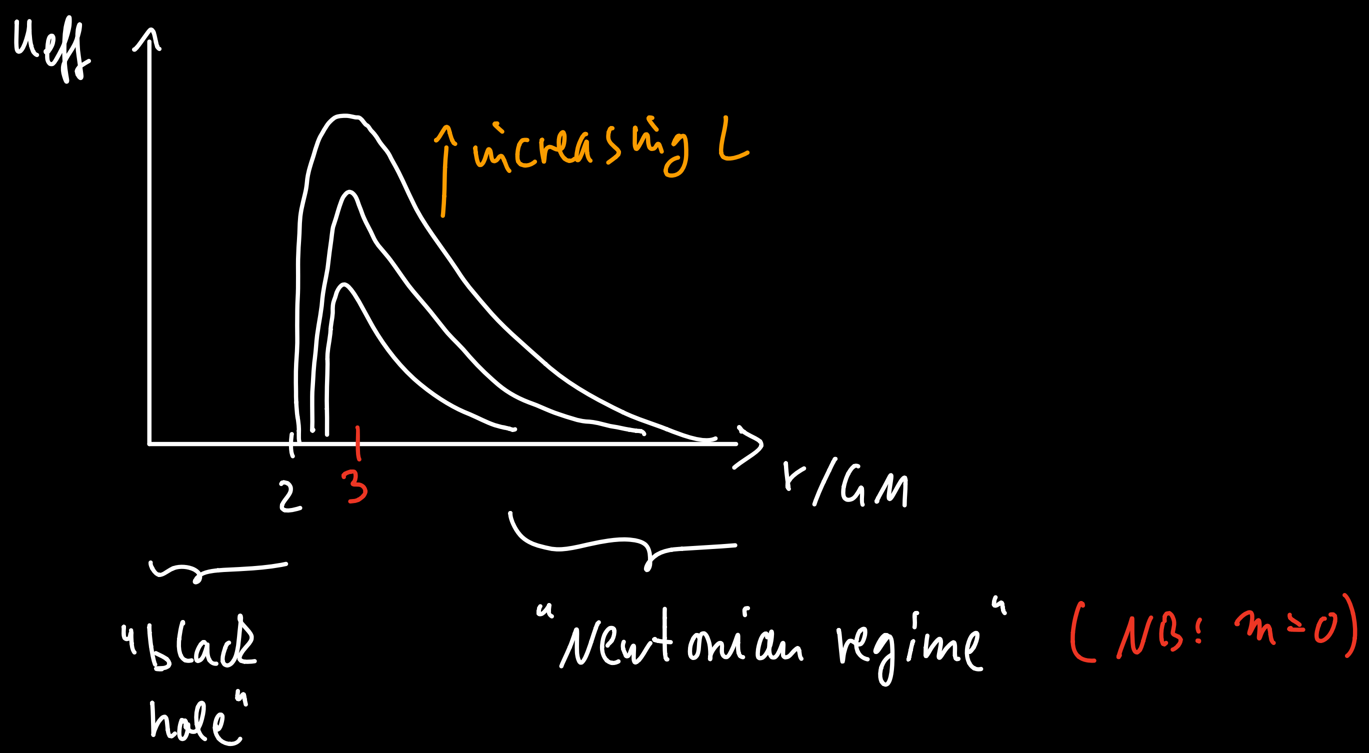
$\Rightarrow$ as before, circular orbits for $U'_{eff} \overset{!}{=} 0$

$$\Leftrightarrow \quad +\varepsilon \frac{GM}{r_c^2} - \frac{L^2}{r_c^3} + \frac{3GML^2}{r_c^4} = 0$$

$$\Leftrightarrow \quad \varepsilon GM r_c^2 - L^2 r_c + 3GML^2 = 0$$

a) massless particles

$\varepsilon = 0 \Rightarrow$ $\boxed{r_c = 3GM}$: unstable circular orbits



b) massive particles

$$\varepsilon = 1 \Rightarrow r_c = \frac{L^2}{2GM}\left(1 \pm \sqrt{1 - 12G^2M^2/L^2}\right)$$

i) large $L \Rightarrow \sqrt{\ } = 1 - 6G^2M^2/L^2$

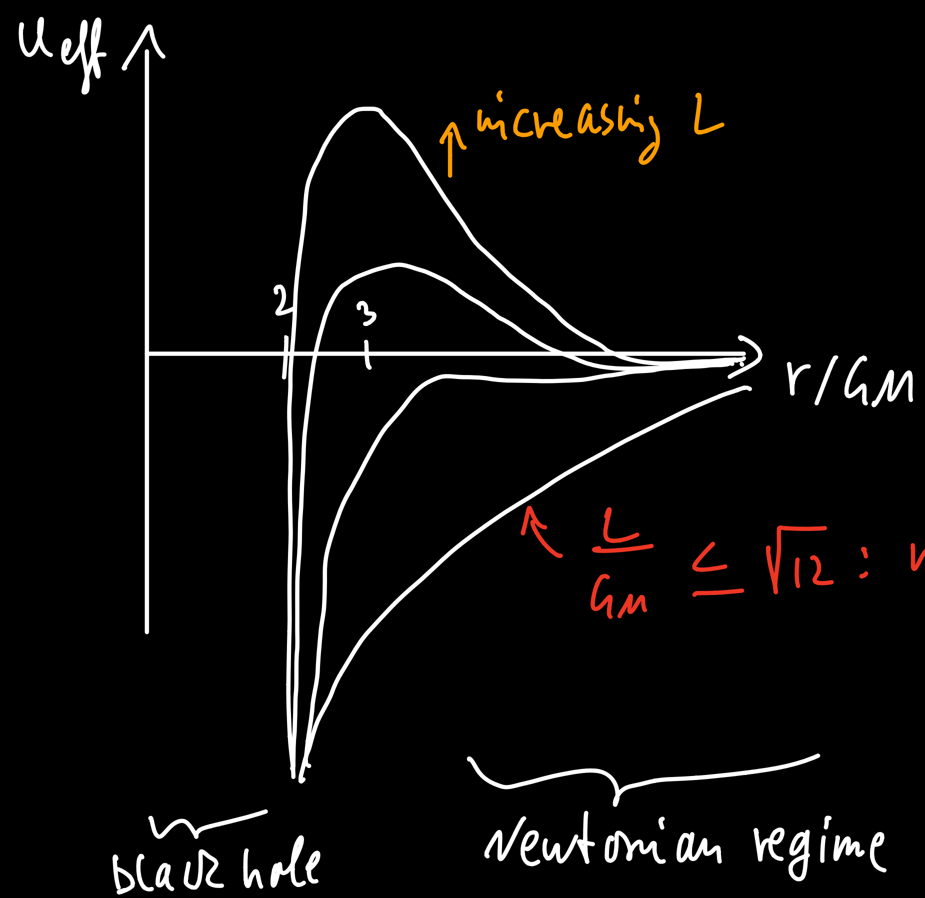
$$\Rightarrow r_c = \left(3GM, \frac{L^2}{GM} \right)$$

unstable ≡ $m=0$ case (for $3GM$); stable orbit ≡ Newtonian case (for $\frac{L^2}{GM}$)

ii) smallest circular orbit: $\sqrt{\ } = 0$

$$\Rightarrow L = \sqrt{12}\, GM \Rightarrow \boxed{r_c = 6GM}$$

- bound, but non-circular orbits oscillate around r_c
  - → Newtonian case: ellipses
  - GR " : " with perihelion precession
    - ⇝ discuss later



5.3 Schwarzschild black holes

$$ds^2 = -\left(1-\frac{R_s}{r}\right)dt^2 + \left(1-\frac{R_s}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$$

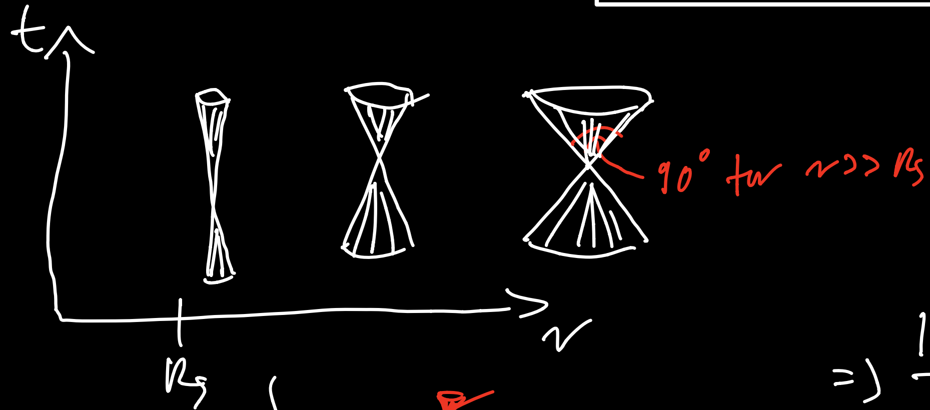
consider radial ($\rightsquigarrow d\Omega = 0$) null curves ($\rightsquigarrow ds^2 = 0$):

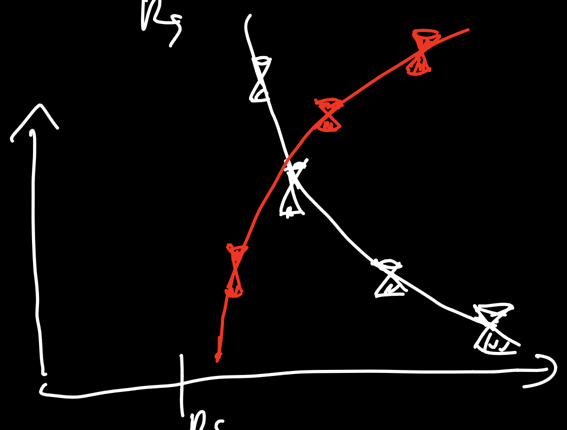
$$0 = -\left(1-\frac{R_s}{r}\right)dt^2 + \left(1-\frac{R_s}{r}\right)^{-1}dr^2$$

$$\Rightarrow \frac{dt}{dr} = \pm\left(1-\frac{R_s}{r}\right)^{-1} = \begin{cases} \pm 1 & \text{for } r \gg R_s \\ \pm\infty & \text{for } r \to R_s \end{cases}$$

$$\Rightarrow t = \text{const.} \pm r^*$$

$$\boxed{r^* \equiv r + R_s \ln\left(\frac{r}{R_s} - 1\right)}$$





$\Rightarrow$ <u>In these co-ordinates</u>, a light ray from $r \gg R_s$ never seems to reach $r = R_s$!

$\Rightarrow$ we can never <u>see</u> an infalling observer reach $r = R_s$!

Q: Do Schwarzschild coordinates cover the whole manifold?

$\rightsquigarrow$ idea: choose alternative time-like coordinate:

$$t \to u \equiv t - r^*$$
$$v \equiv t + r^*$$

"Eddington-Finkelstein coordinates"

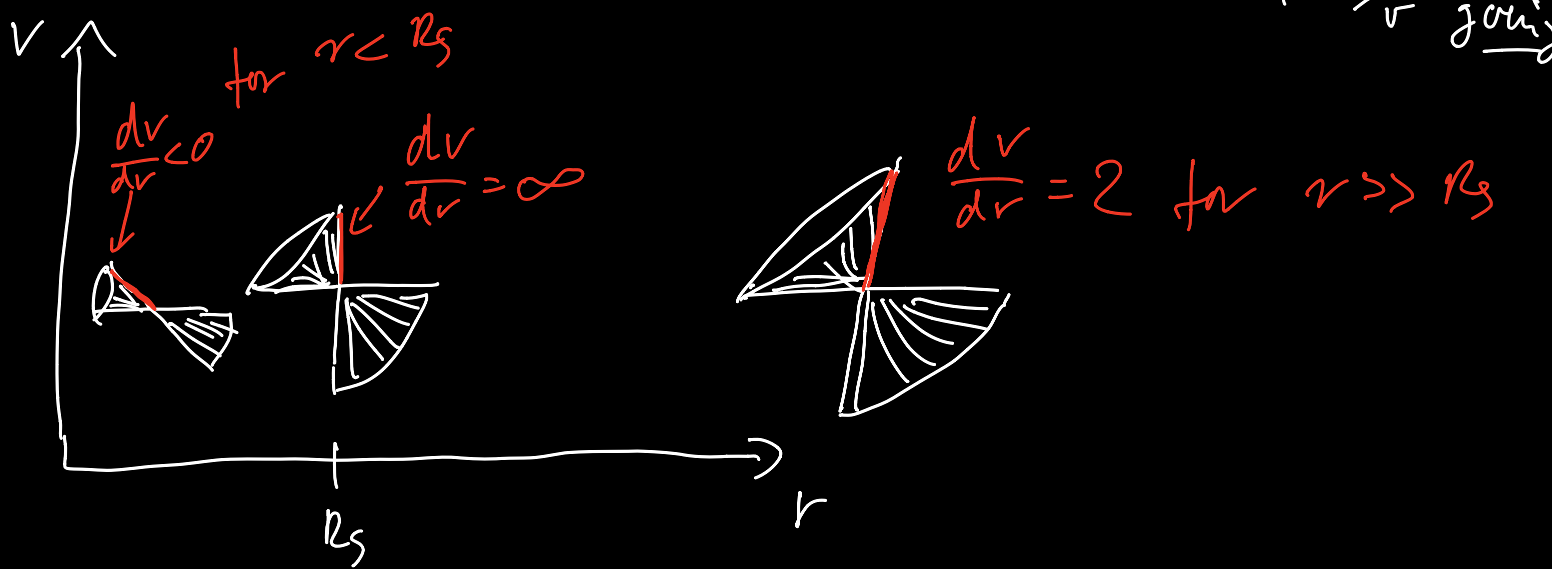
$$\leadsto \dots \quad \boxed{ds^2 = -\left(1-\frac{R_s}{r}\right)dv^2 + 2\,dv\,dr + r^2\,d\Omega^2}$$

NB: $g_{vv} = -\left(1-\frac{R_s}{r}\right) \to 0$ for $r \to R_s$

but $g = \underbrace{\det\begin{pmatrix} -\left(1-\frac{R_s}{r}\right) & 1 \\ 1 & 0 \end{pmatrix}}_{-1} \cdot r^4 \times \sin^2\theta \neq 0$ for $r = R_s$!

radial null curves:

$$0 = -\left(1-\frac{R_s}{r}\right)\left(\frac{dv}{dr}\right)^2 + 2\left(\frac{dv}{dr}\right) \Rightarrow \frac{dv}{dr} = \begin{cases} 0 & : \underline{\text{ingoing}} \\ \dfrac{2}{1-R_s/r} & : \underline{\text{outgoing}} \end{cases}$$



$\Rightarrow$ conclusions: a) "new" part of manifold discovered by following future-directed null geodesics

b) for $r < R_s = 2GM$, $\underline{\text{all}}$ future-directed paths are in the direction of decreasing r!

NB: conclusion independent of coordinate choice because ds^2 = time-like/space-like is independent of coordinate-choice!

$\Leftrightarrow$ impossible to see inside R_s: $\boxed{\text{"black hole"}}$

technical def. of "event horizon"

$:\Leftrightarrow$ surface past which nothing can escape to infinity

NB: global concept!

Q: Are there further regions of the full manifold?

A: yes, can be reached by following

i) past-directed geodesics [choose $v \to u$]

ii) space-like geodesics

Q: $\exists$ global coordinate system to describe the entire manifold?

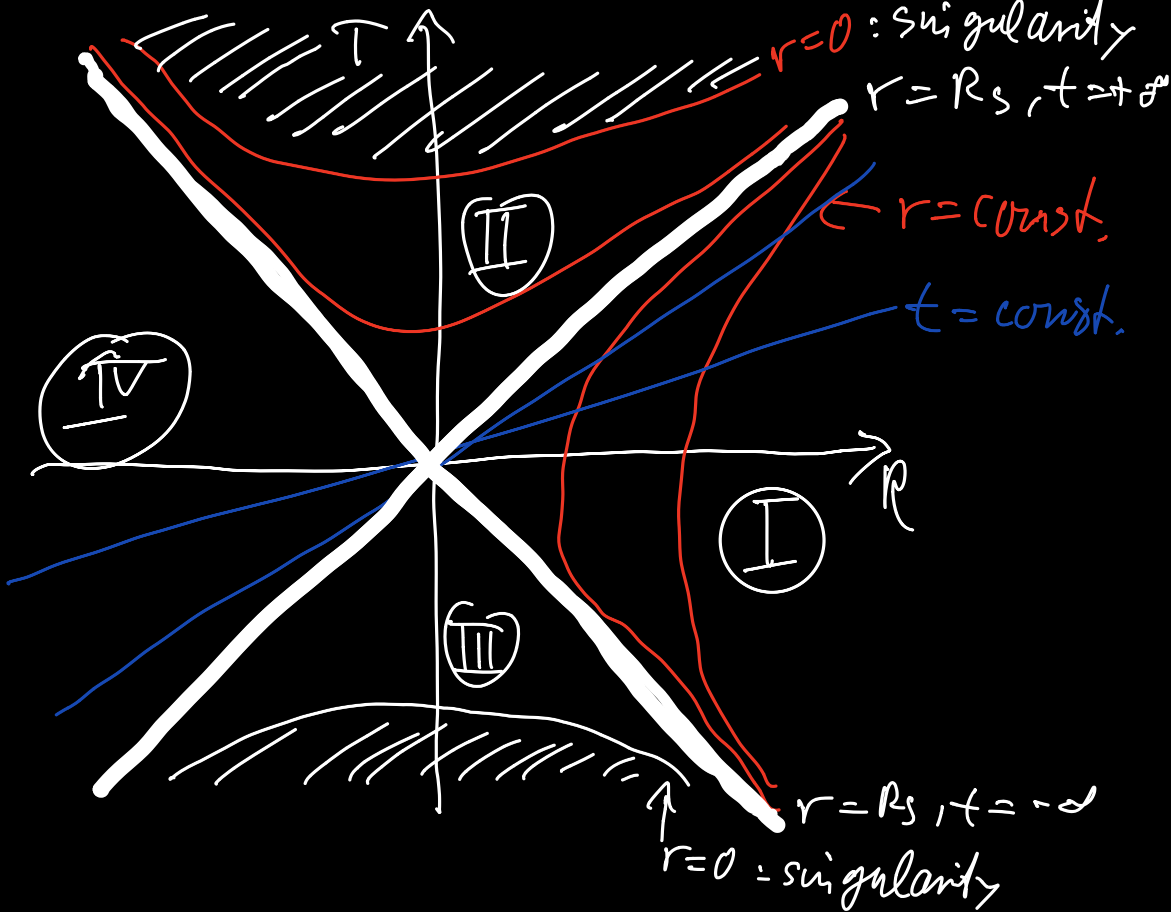
A: yes, Kruskal coordinates

$$T, R \equiv \frac{1}{2}\left(e^{\frac{v}{2R_s}} \mp e^{-\frac{u}{2R_s}}\right)$$

$$\Rightarrow \ldots \quad \boxed{ds^2 = \frac{4R_s^3}{r} e^{-\frac{r}{R_s}}\left(-dT^2 + dR^2\right) + r^2 d\Omega}$$

where $r = r(T,R)$ defined by $T^2 - R^2 \equiv \left(1 - \frac{r}{R_s}\right) e^{\frac{r}{R_s}}$

NB: $ds^2 = 0$ & $d\Omega = 0$ $\Rightarrow \frac{dT}{dR} = \pm 1$: (radial) light cones everywhere at $\pm 45°$!



Ⓘ : $r > R_s$: "our universe"

Ⓘⓘ : black hole : <u>every</u> future-directed path will hit the singularity at $r=0$

Ⓘⓘⓘ : "white hole" : bounded by "past event horizon"

→ everything in Ⓘ seems to spring from past singularity

Ⓘⓥ : "mirror universe", causally disconnected from Ⓘ

Conformal / "Penrose" diagrams

→ powerful way of representing both global properties and causal structure of (sufficiently symmetric) spacetimes

main idea:
- choose coordinates where lightcones have ±45°
- perform a "conformal transformation":

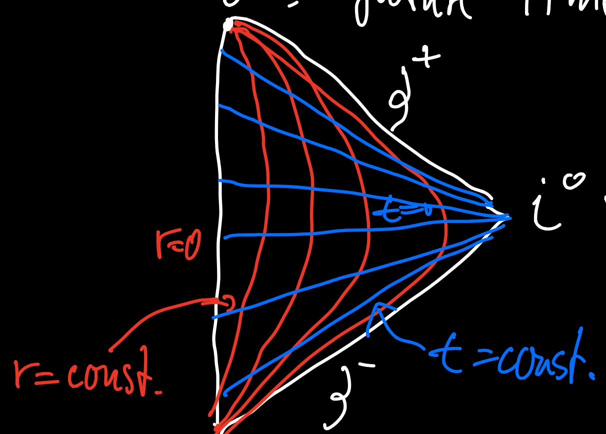
$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \omega^2(x)\, g_{\mu\nu}$$

⇝ light cones are unaffected ($ds^2 = 0$)

- choose $\omega(x)$ such that entire manifold is described by a finite range of coordinates.

e.g. a) all of Minkowski (every point is a 2-sphere)

i^+: "future time-like infinity"



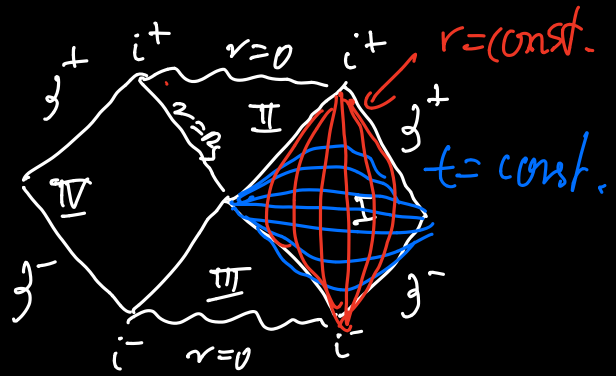
i^-: "past time-like infinity"

$\mathscr{I}^\pm$: future / past null infinity

$\Rightarrow$ all time-like geodesics begin/end at i^-/i^+

〃 null 〃 〃 〃 〃 〃 $\mathscr{I}^-/\mathscr{I}^+$

b) Schwarzschild spacetime:



black hole: singularity hidden behind the event horizon

white hole: "naked" singularity $\Leftrightarrow$ not hidden behind an event horizon

"cosmic censorship conjecture"

~ naked singularities cannot form in gravitational collapse from a "generic", asymptotically flat spacetime.