UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Exam in: General Relativity (FYS4160)

Day of exam: June 14, 2023

Exam hours: 4 hours

This examination paper consists of 4 pages. (including the title page)

Appendices: none

Permitted materials: 3 A4 pages (two-sided) with own notes.

Make sure that your copy of this examination paper is complete before answering.

Final exam

Lecture spring 2023: General Relativity (FYS4160)

⇝ *Carefully **read all questions** before you start to answer them! Note that you do not have to answer the problems in the order presented here, so try to answer those first that you feel most sure about. In particular, questions marked with an* **asterisk** *(∗) require somewhat more heavy calculations / algebra – so once you feel you are getting stuck make sure to move on (and get back to these later).*

Also note that you can (largely) solve each of the subproblems even if you haven't managed to solve the previous subproblems – but this requires to use the information explicitly stated in the previous subproblems. Keep your descriptions as short and concise as possible! Answers given in English are preferred, but feel free to write in Norwegian if you struggle with formulations! Maximal number of available points: 40.

Good luck !!!

Problem 1 *(6 points)*

a) State the equation of motion of a force-free test particle in special and general relativity, in arbitrary coordinates, and discuss the difference! How would external forces appear in these equations? *(3 points)*

b) Consider a spacecraft on the innermost stable circular orbit around a black hole, that ejects a device to measure g-forces. What acceleration does this device measure in the moment where it passes the event horizon? *(1 point)*

c) Consider an observer with 4-velocity U^μ close to a Kerr black hole. Show that this observer measures the speed v of an object with 4-velocity V^μ to be *(2 points)*

$$v = \sqrt{1 - (U_\mu V^\mu)^{-2}}. \tag{1}$$

[Hint: Once you realize that there is a common theme to all of these problems, you also realize that you may not actually need all information that is stated in order to answer some of the questions.]

Problem 2 *(17 points)*

In a Friedman-Robertson-Waker spacetime, the line element can be written as

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a^2(t)\left[\frac{dr^2}{1-\kappa r^2} + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right]. \tag{2}$$

a) What type of coordinates are used here? State this line element instead in Cartesian coordinates, using the fact that the spatial curvature of the universe is observationally determined to be very close to zero. Why can we conclude from this observation that spatial curvature can also be neglected at earlier times? *(3 points)*

b) Show that the only non-vanishing Christoffel symbols are given by $\Gamma^0_{ij} = a^2 u(a)\delta_{ij}$ and $\Gamma^i_{j0} = \Gamma^i_{0j} = u(a)\delta^i_j$, stating the function $u(a)$ explicitly ! *(5 points)*

c) Locally, one can always choose coordinates ξ^μ such that the line element is that of flat space, i.e. $ds^2 = \eta_{\mu\nu} d\xi^\mu d\xi^\nu$ in Cartesian coordinates – or $ds^2 = -(d\xi^0)^2 + (d\xi^r)^2 + (\xi^r)^2 \left[(d\xi^\theta)^2 + \sin^2 \xi^\theta (d\xi^\phi)^2\right]$ in polar coordinates. Show that the coordinate transformation relating the free-fall coordinates to the flat FRW metric is of the form $\partial\xi^0/\partial x^0 = v(a)$, $\partial\xi^i/\partial x^j = w(a)\delta^i_j$ and state $v(a)$ and $w(a)$ explicitly (you can choose either Cartesian or polar coordinates for this).
How does thus the 'physical momentum' $p^i \equiv m\, d\xi^i/d\tau$ of some particle with mass m, as measured by a freely-falling observer, relate to the 'coordinate momentum' $\bar{p}^i \equiv m\, dx^i/d\tau$ observed in the cosmic rest frame ? *(4 points)*

d)* The phase-space distribution $f(\xi^\mu, p^i)$ of some particle species is typically stated in terms of local (free-fall) coordinates and the conjugate momenta. The Liouville operator $L[f]$ that appears on the l.h.s. of the Boltzmann equation in general takes the form $L[f] = df/d\tau$. Evaluate it for the case of a flat FRW spacetime! Using the result for $L[f]$, show that the co-moving number density of a non-interacting collection of particles is conserved. *(5 points)*
[Hint: Start by expanding the total derivative, then change variables from p to $\bar{p}$ (while still treating f as an explicit function of t and p^i). For the next step you will find the result from b) useful. Finally, convert everything back to p^i. NB: You can (almost) completely solve this problem even if you did not find the functional form of u, v and w] !

Problem 3 *(17 points)*
In this problem we investigate how a gravitational wave (GW) changes the frequency ω_γ of a laser in the presence of a gravitational wave with frequency ω_g. For simplicity we consider the effect on an individual photon, and compare the frequency measured by two different observers; let us call them the source (S) and the detector (D). We align the axes of our coordinate system such that the photon initially propagates in the direction of the positive x-axis, i.e. it has an initial 4-momentum $p^\mu|_{t=0} = (\omega_0, \omega_0, 0, 0)$ in the frame of a freely falling observer.

a) In a general frame with metric $g_{\mu\nu}$, what is the photon frequency ω_γ that an observer moving with 4-velocity u^μ measures? Now consider the situation of tiny perturbations to the metric by a GW that passes through, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, inducing both small perturbations δp^μ to the photon momentum and small perturbations δu^μ to the 4-velocities of observers initially at rest (with respect to a freely falling frame). State ω_γ to leading order in the small perturbations! *(4 points)*

b) You should have found that it is only the time-component of the momentum perturbation, δp^0, that enters in the expression for ω_γ. Show that it, to leading order, satisfies

$$\delta p^0 = A \int_0^{\lambda_D} d\lambda' \left[\Gamma^0_{00} + 2\Gamma^0_{10} + \Gamma^0_{11}\right]_{x^\mu = x^\mu_{\lambda',0}}, \tag{3}$$

where $\Gamma^\rho_{\mu\nu}$ denote the Christoffel symbols and λ is the affine parameter that parameterizes the photon geodesic, with $\lambda = 0$ corresponding to $t = 0$ and λ_D being the value of λ at the spacetime point where the photon is detected. Further, we defined $x^\mu_{\lambda,0} \equiv (\lambda\omega_0, \lambda\omega_0, 0, 0)$. Determine the constant A ! *(3 points)*

c)* Now assume that we can choose coordinates such that S and D remain at rest even in the presence of the GW (this is automatically satisfied for freely falling observers in the TT gauge, to be studied below). Put the above above results together to show that the observed frequency shift in that case is given by

$$\frac{\omega_\gamma^D - \omega_\gamma^S}{\omega_\gamma^D} = B \int_0^{\lambda_D} d\lambda' \, \partial_0 \left[h_{00} + 2h_{10} + h_{11}\right]_{x^\mu = x^\mu_{\lambda',0}}, \tag{4}$$

where ω_γ^S is the frequency with which the photon is emitted by the source S at $t = 0$, and ω_γ^D is its frequency as measured by D. Determine the value of the constant B ! *(5 points)*
[Hint: You will encounter an integral over a function $f(x, y)$ that is of the form $\int d\lambda \partial_y f(x(\lambda), y(\lambda))$; use partial integration to bring it into a form that only involves f and $\partial_x f$!]

d) The TT gauge is defined by $h^{\rm TT}_{\mu 0} = \eta^{\mu\nu} h^{\rm TT}_{\mu\nu} = \partial^\mu h^{\rm TT}_{\mu\nu} = 0$. In this gauge, the equation of motion for a GW is given by $\Box h^{\rm TT}_{\mu\nu} = 0$. Write down the general solution of this equation for a plane GW with frequency ω_g that is propagating in x^2 (i.e. 'y') direction, in TT gauge, and compute the frequency shift that is observed at a detector being located a distance L away from the sender.
(5 points)

Useful formulae

$$\Gamma^\mu_{\rho\sigma} = \frac{1}{2} g^{\mu\nu} \left(g_{\rho\nu,\sigma} + g_{\nu\sigma,\rho} - g_{\rho\sigma,\nu}\right) \tag{5}$$

$$\delta^\mu_{\ \nu} = \begin{cases} 1 & \text{if } \mu = \nu \\ 0 & \text{if } \mu \neq \nu \end{cases} \tag{6}$$