i) 1st Friedmann eq. $\boxed{H^2 \equiv \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho_{tot} - \frac{\varkappa}{a^2}}$

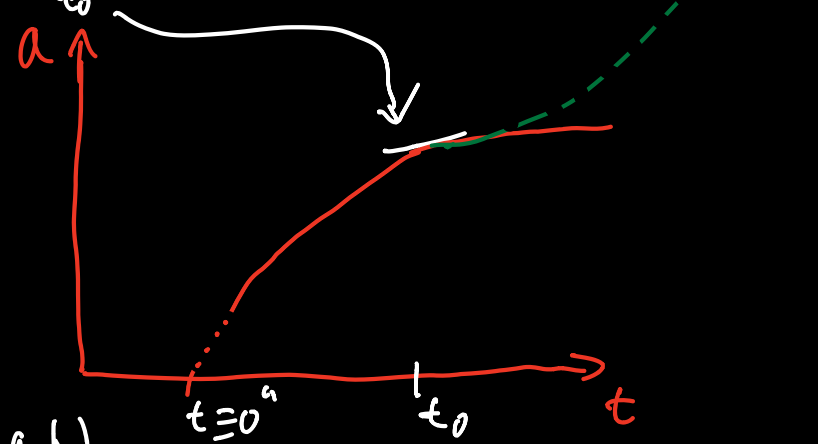
ii) 2nd Friedmann eq. $\boxed{\frac{\ddot a}{a} = -\frac{4\pi G}{3}\left(\rho_{tot} + 3p_{tot}\right)}$

$\rightsquigarrow$ a) $\rho_{tot} + 3p_{tot} > 0 \quad \Rightarrow \quad \ddot a < 0$

"strong energy condition"

b) $\dot a|_{t_0} > 0$ (Hubble)

$\Rightarrow$



a,b) $\Rightarrow a(t_1) < a(t_2) \quad \forall\, t_1 < t_2$

$\Rightarrow$ universe "started" in a "big bang" ($a \sim 0$) not $a = 0$

a finite time ago

$\Rightarrow$ can use $a(t=0) \equiv 0$ to formally define $t = 0$

|| $\not\Rightarrow$ • $a = 0$ ever existed
• $t \leq 0$ impossible ||

$\Rightarrow$ existence of a maximal length that any particle, or piece of information, can have propagated since $t=0$:

"particle horizon": $$\boxed{d_H(t) \equiv a(t)\int_0^{r(t)} \frac{dr'}{\sqrt{1-\kappa r'^2}} \overset{ds^2=0}{=} a(t)\int_0^{t} \frac{dt'}{a}}$$

typically: $H^{-1} \sim d_H$ $\Rightarrow$ $H^{-1} =$ Hubble "horizon"

↑ but not e.g. during inflation!

solving the Friedmann equations

⇝ 2 equations for a, ρ, p

⇝ need 3rd eq ⇝ "equation of state": $p = w\rho$

"eq. of state parameter"

[i) & ii)] or $\nabla_\mu T^\mu{}_0 = 0$

$$\Rightarrow \ldots \quad \dot{\rho} = -3\frac{\dot{a}}{a}(\rho + p)$$

$$\Rightarrow \quad \frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a}$$

if $w = \text{const.}$ $\Rightarrow$ $\rho \propto a^{-3(1+w)}$

$\Rightarrow$ time evolution from Friedmann I:

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho \propto a^{-3(1+w)}$$

$$\Rightarrow \quad a^{\frac{1}{2}+\frac{3}{2}w}\frac{da}{dt} = \text{const.}$$

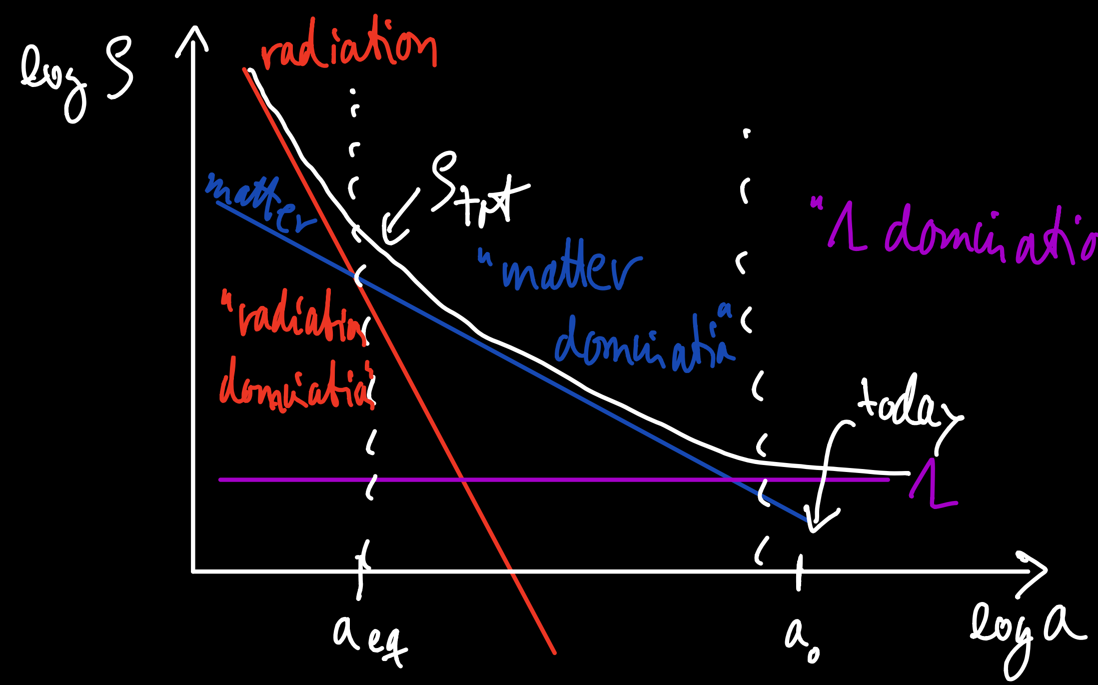
$$\Rightarrow \quad t \propto a^{\frac{3}{2}(1+w)} \quad \Leftrightarrow \quad \boxed{a \propto t^{\frac{2}{3(1+w)}}}$$

	w	ρ	a
matter/ "dust"	0	$\boxed{a^{-3}}$	$t^{2/3}$
radiation	$1/3$	$\boxed{a^{-4}}$	$t^{1/2}$
Λ	-1	const.	e^{Ht}

← expected: dilution of the number density with a volume factor

← additional factor of a: redshift due to expansion of universe!

← de Sitter solution $\left[\left(\frac{\dot a}{a}\right)^2 = \text{const.} = H^2\right]$



$\Rightarrow$ the universe was radiation dominated in the past,
then matter dominated,
then vacuum energy dominated

today : $\Omega_\Lambda \sim 0.7$

$\Omega_m \sim 0.3$

$\Omega_r \sim 10^{-4} \simeq \Omega_{CMB} + \Omega_\nu$

$$\Rightarrow \quad \frac{\rho_m}{\rho_r}\Big|_{t_0} = \frac{a^3 \rho_m}{a^4 \rho_r}\Big|_{t_{eq}} = \frac{1}{a_{eq}} = \frac{\Omega_m}{\Omega_r}\Big|_{t_0}$$

$$\Leftrightarrow \quad \boxed{a_{eq} \sim 3 \cdot 10^{-4}}$$

: scale factor ~~"size" of the universe~~ at the time of matter-radiation equality

⌜ "size of universe" ⇝ size of the visible universe $= d_H$

roughly $d_H \sim H^{-1}$ ⌟

8.3. Thermal history of the early universe

thermodynamics

in equilibrium : $f_i(\vec{p}) = \dfrac{1}{e^{E_i/T} \pm 1}$

+ $\mu = 0$: no chemical potentials

+ : Fermions "Fermi-Dirac statist"

− : Bosons "Bose-Einstein statistic"

$\Rightarrow$ number density : $n_i = g_i \int \frac{d^3p}{(2\pi)^3} f_i(\vec{p})$

$$\rho_i = \langle E_i \rangle \cdot n_i = g_i \int \frac{d^3p}{(2\pi)^3} E_i f_i(\vec{p})$$

pressure of particle species i

$$p_i = \left\langle \frac{\vec{p}_i^{\,2}}{3E_i} \right\rangle n_i = g_i \int \frac{d^3p}{(2\pi)^3} \frac{\vec{p}_i^{\,2}}{3E_i} f_i(\vec{p})$$

two analytic limits:

1) $T \ll m$ (non-rel.) : $n_i = g_i \left(\frac{mT}{2\pi}\right)^{3/2} e^{-m/T}$ Boltzmann suppression

$$\rho_i = m_i n_i \left(+ \tfrac{3}{2} n_i T + \dots \right)$$

$$p_i = T \cdot n_i \ll \rho_i$$

2) $T \gg m$ (relativistic) : $n_i = \frac{\zeta(3)}{\pi^2} g_i T^3 \times \begin{cases} 1 & \text{for BE} \\ 3/4 & \text{for FD} \end{cases}$

$$\rho_i = \frac{\pi^2}{30} g_i T^4 \times \begin{cases} 1 & \text{for BE} \\ 7/8 & \text{for FD} \end{cases}$$

$$\left[\rightarrow \langle E \rangle \approx \begin{cases} 2.7\,T & \text{for BE} \\ 3.15\,T & \text{for FD} \end{cases} \right]$$

$$p_i = \rho_i / 3$$

$$\Rightarrow \boxed{\rho_r^{(tot)} \equiv \frac{\pi^2}{30} g_{eff} T^4} \; ; \text{ where } g_{eff}^{(T)} \equiv \sum_{i=\text{bosons}} g_i \left(\frac{T_i}{T}\right)^4 + \frac{7}{8} \sum_{j=\text{fermions}} g_j \left(\frac{T_j}{T}\right)^4$$

(*)

[= 106.75 for SM and $T \gg m_t$]

$\Rightarrow$ Friedmann I during radiation domination :

$$H^2 = \frac{8\pi G}{3} \rho_r \overset{(*)}{\sim} \Big(1.66 \underbrace{G^{1/2}}_{1/M_{pl}} \sqrt{g_{eff}}\, T^2\Big)^2$$

$$H = \frac{\dot{a}}{a} = \frac{1}{2t} \quad \Rightarrow \boxed{t \simeq 0.3 \frac{M_{pl}}{\sqrt{g_{eff}}\, T^2} \sim \left(\frac{\text{MeV}}{T}\right)^2 \text{sec}}$$

$\Rightarrow$ "<u>Hot</u> Big Bang"