entropy $\mathcal{S}$

in therm. eq.: $T\,d\mathcal{S} = dE + p\,dV$

$$\Rightarrow d\mathcal{S}(V,T) = \frac{1}{T}\left\{ d\left[V\varepsilon(T)\right] + p(T)\,dV \right\}$$

$$= \underbrace{\frac{\varepsilon+p}{T}}_{=\frac{\partial\mathcal{S}}{\partial V}\ (\mathrm{I})} dV + \underbrace{\frac{V}{T}\frac{d\varepsilon}{dT}}_{=\frac{\partial\mathcal{S}}{\partial T}\ (\mathrm{II})} dT$$

$$\frac{\partial^2\mathcal{S}}{\partial T\,\partial V} \overset{!}{=} \frac{\partial^2\mathcal{S}}{\partial V\,\partial T}$$

$$\Leftrightarrow \frac{\partial}{\partial T}\overset{\mathrm{I}}{\left[\frac{\varepsilon+p}{T}\right]} \overset{!}{=} \frac{\partial}{\partial V}\overset{\mathrm{II}}{\left[\frac{V}{T}\frac{d\varepsilon}{dT}\right]}$$

$$\Leftrightarrow -\frac{1}{T^2}(\varepsilon+p) + \frac{1}{T}\frac{d}{dT}(\cancel{\varepsilon}+p) = \cancel{\frac{1}{T}\frac{d\varepsilon}{dT}} \Rightarrow \underline{\underline{\frac{dp}{dT} = \frac{\varepsilon+p}{T}}}$$

$$\Rightarrow \frac{\partial\mathcal{S}}{\partial T} \overset{\mathrm{II}}{=} \frac{V}{T}\frac{d\varepsilon}{dT} = \frac{V}{T}\frac{d(\varepsilon+p)}{dT} - \frac{V}{T}\frac{dp}{dT} = \underline{\underline{V\frac{d}{dT}\left(\frac{\varepsilon+p}{T}\right)}}\ (\mathrm{III})$$

$$\Rightarrow (\mathrm{III}+\mathrm{I}):\quad \mathcal{S} = V\frac{\varepsilon+p}{T}\quad (+\text{ const.})$$

$$\Rightarrow \boxed{s \equiv \frac{\mathcal{S}}{V} = \frac{\varepsilon+p}{T}}$$

<u>entropy density</u>
(valid in thermal equilibrium)

$$\Rightarrow \frac{d}{dt}(a^3 s) = \frac{1}{T}\frac{d}{dt}(a^3\rho) + \frac{1}{T}\frac{d}{dt}(a^3 p) - \frac{\rho+p}{T^2}a^3\dot{T}$$

$$= \frac{1}{T}\Big\{ \underbrace{\frac{d}{dt}(a^3\rho) + p\frac{d}{dt}a^3}_{=0 \text{ (from } T^{\mu\nu}{}_{;\nu}=0)} + \underbrace{\underbrace{\frac{dp}{dT}}_{=\frac{\rho+p}{T}} a^3\dot{T} - \frac{\rho+p}{T}a^3\dot{T}}_{=0} \Big\}$$

$\Rightarrow$ **entropy conserved in thermal equilibrium!**

"adiabatic approximation"

$$\Rightarrow \boxed{s = \frac{\rho+p}{T} = \frac{2\pi^2}{45} g^s_{\text{eff}} T^3} \quad ; \quad g^s_{\text{eff}} = \sum_{\text{bosons}} g_i\left(\frac{T_i}{T}\right)^3 + \frac{7}{8}\sum_{\text{fermions}} g_j\left(\frac{T_j}{T}\right)^3$$

$T \gg m$

$$\Rightarrow \quad s a^3 = \text{const.} \quad \Rightarrow \quad \boxed{\boxed{T \propto g^{s\,-1/3}_{\text{eff}}\, a^{-1}}} \quad \text{vs "hot big bang"!}$$

What to do outside equilibrium /
how to describe particle species falling out of eq. ?

$\Downarrow$

<u>Boltzmann equation</u>

describes evolution $f(\vec{x}, \vec{p})$

$$\hat{L}[f] = \hat{C}[f]$$

"Liouville operator"

collision term

$$\hat{L}[f] = \frac{df}{d\lambda} = \frac{dx^i}{d\lambda}\frac{\partial f}{\partial x^i} + \frac{dp^i}{d\lambda}\frac{\partial f}{\partial p^i} \quad \Big| \; \lambda = \tau$$

$$= p^i \frac{\partial f}{\partial x^i} - \Gamma^i_{s\sigma} p^s p^\sigma \frac{\partial f}{\partial p^i} \quad \Big| \text{ use } \Gamma^i_{s\sigma} \text{ for FRW}$$

$$\vdots$$

$$= p^0 (\partial_t - H \vec{p} \cdot \nabla_{\vec{p}}) f(t, \vec{p})$$

physical momentum
(not comoving !!)

$$\vec{p} = \frac{\vec{\tilde{p}}}{a}$$

for the number density, we just need to integrate over the momenta:

$$g_i \int \frac{d^3p}{(2\pi)^3} \frac{\hat{L}[f_i]}{p^0} = \partial_t \Big[\underbrace{g_i \int \frac{d^3p}{(2\pi)^3} f_i}_{n_i} \Big] - H g_i \int \frac{d^3p}{(2\pi)^3} \underbrace{\vec{p}\,\nabla_{\vec{p}} f_i}_{\to\, -f_i \underbrace{\nabla_{\vec{p}}\vec{p}}_{\frac{\partial}{\partial p_i} p_i = 3}}$$

$$= \dot{n}_i + 3Hn_i \qquad \Big|\ H = \frac{\dot{a}}{a}$$

$$= \frac{1}{a^3} \frac{d}{dt} (\underbrace{a^3 n_i}_{\text{comoving number density}})$$

$\Rightarrow$ constant for $C[f] = 0$ ✓

freeze-out mechanism

static case: all particle species in equilibrium, e.g. through

$$i + \bar{i} \leftrightarrow k + l$$

expanding universe: competition between

- H - expansion rate
- $\Gamma = n \sigma v$ interaction rate

$\rightarrow$ @ large T: $\Gamma \gg H$: thermal eq.

@ "freeze-out" T_f: $\Gamma \sim H$

@ small T ($T < T_f$): particle species i decoupled from heat bath

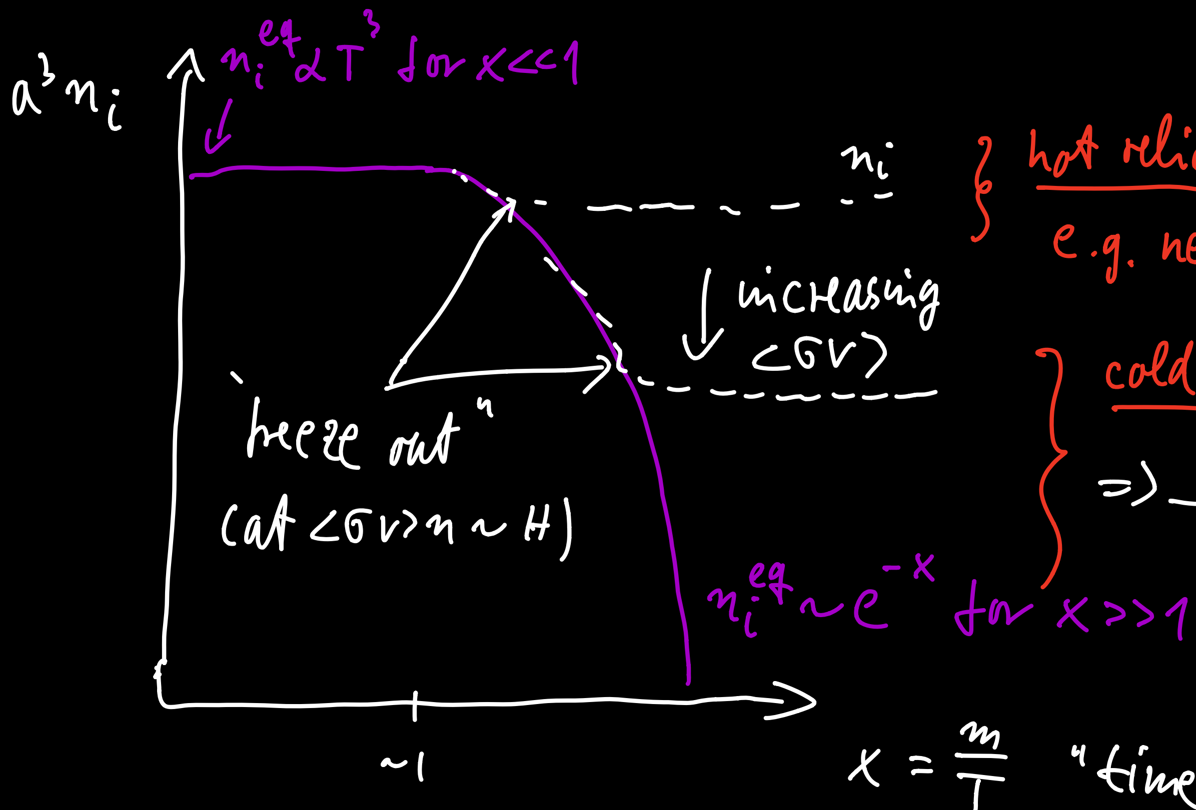
$\Leftrightarrow C[f] = 0$

in detail: $\hat{L}[f_i] = C[f_i]$

$\Big\downarrow \; g_i \int \frac{d^3p}{(2\pi)^3} \frac{1}{E}$

assume $n_i = n_{\bar{i}}$ (no asymmetry)

$$\boxed{\frac{dn_i}{dt} + 3Hn_i = -\langle \sigma v \rangle \left[n_i^2 - (n_i^{eq})^2 \right]}$$



hot relic ($T_f \gtrsim 3m_i$)
e.g. neutrinos

cold relic ($T_f \ll m_i$)

$\Rightarrow \Omega_i \propto \frac{1}{\langle \sigma v \rangle}\Big|_{T_f}$

e.g. dark matter!