Relativistic analysis

[c.f. Weinberg]

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}$$

FRW

use "Hodge decomposition"

e.g. for 3-vector:

$$A^i = \partial^i \phi + B^i$$

"irrotational part" ($\vec{\nabla} \times \nabla\phi = 0$)

w/ $\vec{\nabla}\vec{B} = 0$ "divergence free"

[proof: $\vec{\nabla}\vec{A} = \Delta\phi$ ⇝ poisson eq. for ϕ

⇝ $\vec{B} = \vec{A} - \nabla\phi$]

tensor: additionally separate trace from trace-free part

$$⇝ \boxed{\begin{aligned} g_{\mu\nu}\,dx^\mu dx^\nu = &-[1+E]\,dt^2 + a^2(t)[\underbrace{\partial_i F + G_i}_{=\frac{1}{2}\delta g_{0i}}]\,dt\,dx^i \\ &+ a^2(t)[(1+A)\delta_{ij} + \partial_i\partial_j B + \partial_i C_j + \partial_j C_i + D_{ij}]\,dx^i dx^j \end{aligned}}$$

A, B, E, F : "scalar perturbations"

C_i, G_i : "vector perturbations" with $\vec{\nabla}\vec{C} = \vec{\nabla}G = 0$

↳ cosmologically not relevant (only decaying solutions!)

D_{ij} : "tensor perturbation" (= grav. waves)

with $\partial_i D_{ij} = 0$, $D_{ii} = 0$

$\rightsquigarrow$ not interested in here

similar : $T_{\mu\nu} = \bar{T}_{\mu\nu} + \delta T_{\mu\nu}$ $[= (p+\rho) u_\mu u_\nu + p g_{\mu\nu}]$

$\rho = \bar{\rho} + \delta\rho$

$p = \bar{p} + \delta p$

$u_\nu = (1, \vec{0}) + \partial_\nu \delta u + \delta u^V_\nu$

$\rightsquigarrow$ keep only linear terms for $\delta T_{\mu\nu}$

$\rightsquigarrow \delta T_{ij} = \bar{p}\, \delta g_{ij} + a^2 (\delta_{ij} \delta p + \partial_i \partial_j \pi^S [+ \partial_i \pi_j^V + \partial_j \pi_i^V + \pi_{ij}^T])$

$\delta T_{i0} = \bar{p}\, \delta g_{i0} - (\bar{\rho} + \bar{p})(\partial_i \delta u [+ \delta u_i^V])$

$\delta T_{00} = -\bar{\rho}\, \delta g_{00} + \delta\rho$

dissipative terms

but : $\delta\rho$ etc. are coordinate-dependent!

consider $x^\mu \to x'^\mu = x^\mu + \xi^\mu(x)$ $[\xi^\mu \ll x^\mu]$

$\Rightarrow g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = g_{\rho\sigma}(x) \dfrac{\partial x^\rho}{\partial x'^\mu} \dfrac{\partial x^\sigma}{\partial x'^\nu}$

$$= (\bar g_{\rho\sigma} + \delta g_{\rho\sigma})\left(\delta^\rho_\mu - \frac{\partial \xi^\rho}{\partial x^\mu}\right)\left(\delta^\sigma_\nu - \frac{\partial \xi^\sigma}{\partial x^\nu}\right) + \dots$$

$\left(\frac{\partial}{\partial x'} \to \frac{\partial}{\partial x}\right)$

$$= \bar g_{\mu\nu}(x) + \delta g_{\mu\nu}(x) - \bar g_{\rho\nu}\frac{\partial \xi^\rho}{\partial x^\mu} - \bar g_{\mu\sigma}\frac{\partial \xi^\sigma}{\partial x^\nu} + \dots$$

$$= \bar g_{\mu\nu}(x') + \delta g_{\mu\nu}(x') \underbrace{- \xi^\rho \frac{\partial}{\partial x^\rho}\bar g_{\mu\nu}(x') - \bar g_{\rho\nu}\frac{\partial \xi^\rho}{\partial x^\mu} - \bar g_{\mu\sigma}\frac{\partial \xi^\sigma}{\partial x^\nu}}_{\equiv \Delta(\delta g_{\mu\nu})} + \dots$$

induced by coordinate transformation

similar: $\Delta \delta p = \dot{\bar p}\, \xi_0$

$\Delta \delta \rho = \dot{\bar \rho}\, \xi_0$

$\Delta \delta u = - \xi_0$

(other d.o.f. of $T^{\mu\nu}$ are gauge invariant)

$\Rightarrow$ e.g. $\delta \equiv \frac{\delta \rho}{\bar \rho}$ is gauge-dependent and hence <u>not</u> a physical observable!

popular gauge choices: • "Newtonian gauge"

$B = F = 0, \quad E \equiv 2\phi, \quad A \equiv -2\psi$

⌜ ⇝ if $\pi^S = 0 \Rightarrow$ (field equations) $\psi = \phi$

⇝ for $\lambda \ll aH^{-1}$: (sub-Horizon scales)

$\Delta \psi = 4\pi G \rho$

(i.e. $\psi \hat{=}$ classical Newtonian potential) ⌟

• "comoving"/"total matter" gauge

$F = 0, \ \delta u = 0$

alternative: gauge-invariant approach

e.g. "curvature perturbation" $\boxed{\mathcal{R} \equiv \frac{A}{2} + H\,\delta u}$

⌜ in comoving gauge: $H^2 = \frac{8\pi G}{3}\rho + \frac{2}{3} D^2 \mathcal{R}$

⇝ $\mathcal{R}$ = "spatial curvature as seen by a comoving observer" ⌟

Field equations

\+ super-horizon scales $(k \ll aH)$

\+ "adiabatic perturbations"

$\left(\frac{\delta_i}{\bar{\rho}_i + \bar{p}_i}\right.$ same for all components in comoving gauge)

$\left.\begin{matrix} \\ \\ \\ \end{matrix}\right\}$ $\mathcal{R} = \text{const.}$

$\Rightarrow$ e.g. in • comoving gauge:

$$\mathcal{R}_k = \frac{5+3w}{3+3w}\,\frac{2}{3}\left(\frac{aH}{k}\right)^2 \delta_k$$

$\Rightarrow$ before horizon entry $(k < aH)$:

$$\delta_k \propto (aH)^{-2} \propto \frac{t^2}{a^2} \propto \begin{cases} a^2 & \text{during RD} \\ a & \quad " \quad \text{MD} \end{cases}$$

• Newtonian gauge: $\mathcal{R} \propto \delta$

$\Rightarrow$ all $\delta^i = \text{const.}$ outside horizon!

• makes sense: no causal contact $\rightsquigarrow$ density contrast cannot grow!

• BUT highly gauge dependent! (see above

way to proceed :
- use field equations
- energy/momentum conservation for each fluid component

⋮

- always keep growing solution
  ⌜large decaying mode even inconsistent w/ cosmological principle!⌟

⋮

inside the horizon ($k \gg aH$)

⇝ δ ~ gauge-independent !

$\boxed{t < t_{eq}}$: $\delta_\gamma \simeq \delta_b \propto \cos k r_s$

"sound horizon", $r_s \equiv \int_0^t dt \frac{c_s}{a} \simeq c_s d_H$

$c_s \equiv \sqrt{\frac{dp}{d\rho}} \simeq \frac{1}{\sqrt{3}}$

$\delta_{DM} \propto \log t$

$\boxed{t > t_{eq}}$ $\delta_{DM} \propto a(t) \propto t^{2/3}$

$\boxed{t \geq t_{rec}}$ $\delta_b \longrightarrow \delta_{DM} \propto a(t)$ ⌜baryons fall into gravitational potential provided by DM⌟

γ-b decoupling

↳ CMBR

$\boxed{t \geq t_\Lambda \; (\rho_\Lambda \geq \rho_m)}$ $\delta_b, \delta_{DM} \sim$ const.

<u>NB</u> : CMB measurements : $\delta_b(t_{rec}) \sim 10^{-5}$

assuming no DM : $\delta_b(t_0) \sim \underbrace{a^{-1}(t_{rec})}_{\sim 10^3} \delta_b(t_{rec})$

$\sim 10^{-2}$

$\Rightarrow$ still in linear regime

$\Rightarrow$ <u>no</u> gravitational structures today!

$\leadsto$ very existence of galaxies is "proof" of "dark matter" (= additional gravitational potentials)!