8.5. (A taste of) inflation

problems of Big Bang picture

a) flatness problem

$$\Omega(t) - 1 = \frac{k}{a^2} \propto \begin{cases} a(t) & \text{during MD} \\ a^2(t) & \text{during RD} \end{cases}$$

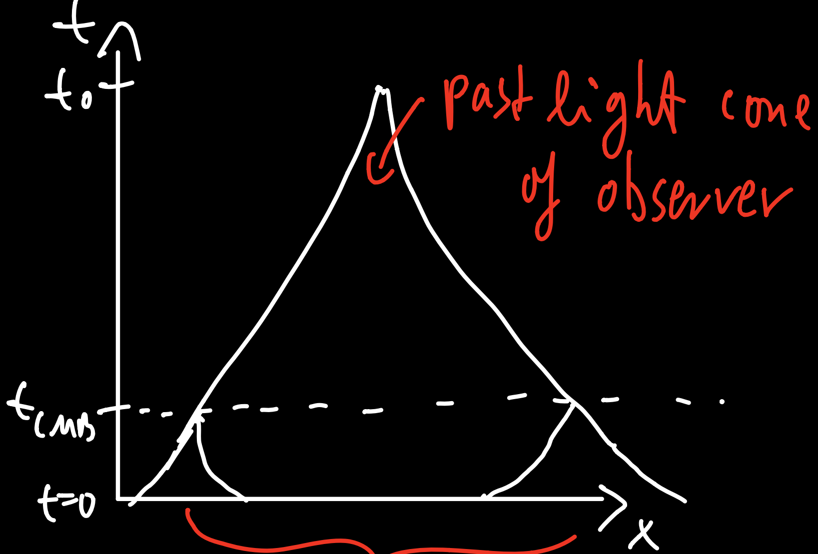
today: $\Omega - 1 \lesssim 10^{-2}$

$\Rightarrow$ $t = 1\,s$: $\Omega - 1 \lesssim 10^{-17}$

$t = 10^{-43}\,s$: $\Omega - 1 \lesssim 10^{-60}$

$\Rightarrow$ requires extreme fine-tuning in initial conditions!

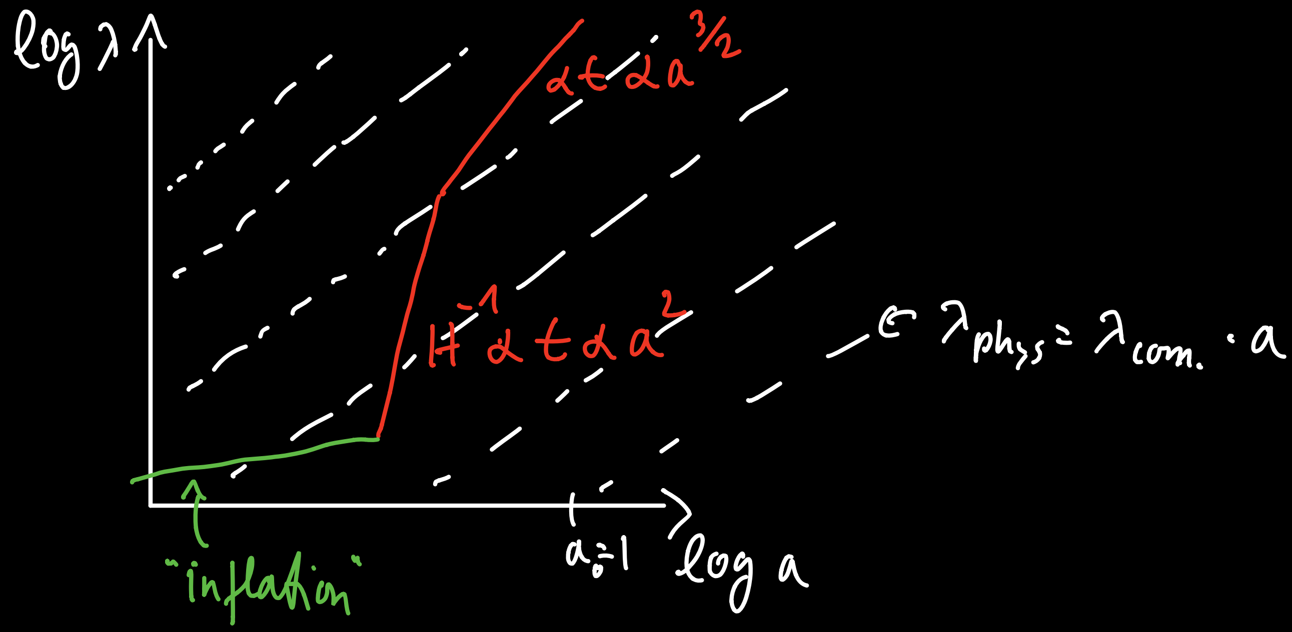
b) horizon problem



no overlap

$\rightarrow$ causally disconnected ($\sim 10^5$)

$\Rightarrow$ why identical $T = 2.73\,K$??



$$H^{-1}(t_{enter}) \equiv a\lambda \iff \frac{k}{2\pi} \equiv aH|_{t_e}$$

"time of horizon entry"

⇝ <u>all</u> scales have been outside the horizon

⇝ why does the universe look the same in different directions??

⌈any physical mechanism responsible for this can only have been active for $t > t_e$⌋

[c) monopole problem]

d) what generated $\delta(t, \vec{x})$?

NB: no causal mechanism possible!

solution

assume early period with decreasing comoving Hubble "horizon":

$$0 > \frac{d}{dt}\frac{1}{aH} = \frac{d}{dt}\frac{1}{\dot a} = -\frac{\ddot a}{(\dot a)^2} \Leftrightarrow \ddot a > 0$$

$$\Leftrightarrow \boxed{\rho + 3p < 0}$$

2nd Friedmann

$\Rightarrow$ for all scales k: two solutions to $k = aH$

$$t > t_R^{\text{enter}} : \text{ inside the horizon}$$
$$t_R^{\text{exit}} < t < t_R^{\text{enter}} : \text{ outside the horizon}$$
$$t < t_R^{\text{exit}} : \text{ inside the horizon}$$

$\rightarrow$ can solve Horizon problem!

$\rightsquigarrow$ coordinate size of particle horizon @ t_{CMB}:

$$\frac{d_H(t_{CMB})}{a} = \int_0^{t_{CMB}} \frac{dt}{a(t)} \simeq \int_0^{t_{eq}} \frac{dt}{a_{eq}\left(\frac{t}{t_{eq}}\right)^{1/2}} \left[+ \int_{t_{eq}}^{t_{CMB}} \frac{dt}{a_{eq}\left(\frac{t}{t_{eq}}\right)^{2/3}} \right] \text{(standard Big Bang)}$$

$$\ll \int_{t_{CMB}}^{t_0} \frac{dt}{a(t)} \sim \frac{1}{[a_0] H_0} \quad (*)$$

$\gg$

with an initial period of inflation:

e.g. $a(t) = a_i \, e^{H_I (t - t_i)}$ for $t_i < t < t_f$; $H_I \simeq$ const.

$$\rightsquigarrow \frac{d_H(t_{CMB})}{a} = \left[\underbrace{\int_0^{t_i} \frac{dt}{a(t)}}_{\overset{?}{\approx} 0} \right] + \underbrace{\int_{t_i}^{t_f} \frac{dt}{a_f \, e^{H_I (t - t_f)}}}_{-\frac{1}{a_f H_I} \underbrace{\left[e^{-H_I (t - t_f)} \right]_{t_i}^{t_f}}_{1 - e^{-H_I (t_i - t_f)} \simeq -\frac{a_f}{a_i}}} \overset{!}{\gg} \left[+ \int_{t_f}^{t_{eq}} \frac{dt}{a_f \left(\frac{t}{t_{eq}} \right)^{1/2}} \right]$$

$\rightsquigarrow$ solving the Horizon problem requires

$$\frac{1}{a_i H_I} \overset{!}{\gtrsim} \frac{1}{a_0 H_0} \quad (*)$$

$$\Leftrightarrow \quad \frac{a_f}{a_i} \gtrsim \frac{a_f H_I}{a_0 H_0}$$

$\longrightarrow a_f H_f$; $aH \propto t^{-1/2} \propto a^{-1}$ (RD)

$\Rightarrow (a_f H_I) \sim (a_0 H_0) \cdot \frac{a_0}{a_f}$

neglect difference between MD and RD here

$$\Rightarrow \boxed{\frac{a_f}{a_i} \gtrsim \frac{a_0}{a_f}} \quad \text{"inflation"}$$

i.e. horizon problem is solved if growth during inflation equals growth afterwards

$$\frac{a_0}{a_f} \sim \begin{cases} e^{60} & \text{for } T_f \sim 10^{16}\ \text{GeV} \\ e^{30} & \text{for } T_f \sim 1\ \text{TeV} \end{cases}$$

$\Rightarrow$ solving the horizon problem requires

$\gtrsim 30-60$ "e-folds", $N \equiv \ln \frac{a_f}{a_i}$

mechanism

- How to get $a \propto e^{Ht}$?

  $\hookrightarrow$ vacuum energy! $\rho = \frac{\Lambda}{8\pi G} = -p$

$$\left(\frac{\dot{a}}{a}\right)^2 = H_I^2 = \frac{8\pi G}{3}\rho = \text{const.}$$

$$\Rightarrow a \propto e^{H_I t} \quad \text{"de Sitter space"}$$

- but need to start/end as well!

  $\Lambda \rightarrow$ scalar field ϕ

$$\mathcal{L}_\phi = \frac{1}{2}\partial^\mu\phi\,\partial_\mu\phi - V(\phi) + \mathcal{L}_{\phi,SM}$$

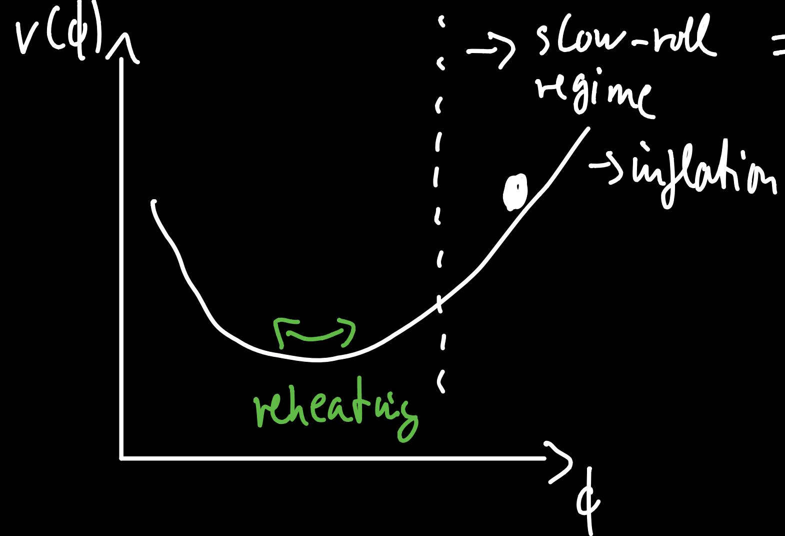
$$\Rightarrow T^{\mu\nu} = \frac{\delta}{\delta g^{\mu\nu}}\left(\sqrt{-g}\,\mathcal{L}_\phi\right)$$

FRW

$$\Rightarrow \dots \quad \rho_\phi = T^{00} = \frac{1}{2}\dot{\phi}^2 + V(\phi) \left[+ \frac{(\nabla\phi)^2}{2a^2} \right] \sim \text{homogeneous}$$

$$p_\phi = \frac{1}{3} T^i{}_i = \frac{1}{2}\dot{\phi}^2 \underbrace{- V(\phi)}_{} \left[- \frac{(\nabla\phi)^2}{6a} \right]$$

dominates during "slow roll": $\frac{1}{2}\dot{\phi}^2 \ll |V(\phi)|$



$\Rightarrow$ ϕ behaves roughly as Λ!

$\delta \int d^4x \sqrt{g}\, \mathcal{L}_\phi \doteq 0 \Rightarrow$ Klein-Gordon equation:

$$0 = \Box \phi + V'(\phi)$$

$$\Box = \nabla^\mu \nabla_\mu \phi$$

$$\overset{\text{FRW}}{=} \ddot{\phi} + (3H\,{+\Gamma})\,\dot{\phi} \left[+ \nabla^2 \phi \right] + V'(\phi)$$

Γ: decay rate for $\phi \to$ SM particles

$\rightsquigarrow$ "reheating" $\to$ standard RD!

density perturbations

classical : $\phi(t,\vec{x}) = \phi(t) + \delta\phi(t,\vec{x})$

dominates $T^{\mu\nu}$, drives inflation, reheats universe

leads to density fluctuations

insert in KG, keep only first order terms

...

$$0 = \delta\ddot{\phi}_k + \underbrace{3H\,\delta\dot{\phi}_k}_{\text{dominates for } t > t^k_{exit}} + \underbrace{\left(\frac{k}{a}\right)^2\delta\phi_k + m^2\delta\phi_k}_{\text{dominates for } t < t^k_{exit}}$$

$\rightsquigarrow$ GR pert. theory ...

$$\mathcal{R}_k \simeq -\frac{H}{\dot{\phi}}\delta\phi_k\Big|_{t \geq t_{exit}}$$

(= const. on super-horizon scales)

quantum description : $\delta\phi_k$ (field) $\longrightarrow$ $\delta\hat{\phi}_k$ (operator)

$$\delta\hat{\phi}_k = w_k(t)\,a_k + w_k^*(t)\,a^\dagger_{-k} \;;$$

$$[a_k, a^\dagger_{k'}] = \delta^{(3)}(\vec{k}-\vec{k}')$$

$$\Rightarrow \langle 0 | \, |\delta\hat{\phi}_k|^2 \, |0\rangle = \langle 0 | (w_k a_k + w_k^* a_{-k}^\dagger)(w_k^* a_k^\dagger + w_k a_{-k}) |0\rangle$$

$$a_k |0\rangle = 0$$

$$= |w_k|^2$$

$$\propto \left. \frac{H^2}{k^3} \right|_{t_{exit}}$$

H should be ~constant
$\Rightarrow$ weak dependence on k
$\Rightarrow$ characteristic spectrum (prediction!)

$\neq 0$! i.e particle / mode production during expansion

during expansion:

$$\langle 0 | \, |\delta\hat{\phi}_k|^2 \, |0\rangle \longrightarrow \langle (\delta\phi_k)^2 \rangle$$

classical ensemble

$\Rightarrow$ assuming "nothing" (= vacuum state) as initial condition + an early period of accelerated expansion (inflation) can explain the entire observable universe!