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http://www.uio.no/studier/emner/matnat/fys/FYS4160/v24/

Lecture spring 2024:

General Relativity

Problem sheet 1

⇝ These problems will be discussed on **Thursday, 1 February 2024**.

Problem 3

In special relativity, time passes slower in a moving frame than in the rest frame:

a) Consider a clock that moves with speed v with respect to an observer. Derive the 'time-dilation factor' between the time that the moving clock shows and the time that the observer measures in its rest frame, using *i)* a spacetime diagram that includes the worldlines of the two observers and *ii)* the explicit form of a Lorentz boost as derived in the lecture.

b) Most of the high-energy secondary cosmic rays that reach the earth are muons, which decay with a lifetime of $\tau_0 = 2.2 \times 10^{-6}\,\mathrm{s}$ into electrons and (anti-) neutrinos, $\mu^\pm \to e^\pm + \nu_\mu + \bar{\nu}_e$. Assume that the muons are produced at an altitude of $h = 10\,\mathrm{km}$ and move downwards, perpendicular to the surface of the earth, with an energy of $E = 1\,\mathrm{GeV}$. Given a mass of $m_\mu = 106\,\mathrm{MeV}$, what does that imply for their velocity in terms of c? Discuss the flux ratio Φ_h/Φ_0, where Φ_h is the flux at $h = 10\,\mathrm{km}$ and Φ_0 the flux at ground, both in the muon rest frame and in the laboratory system (i.e. the surface of the earth)!
[Hint: The conclusion should be that the concepts of 'length contraction' and 'time dilation' are indeed the same thing, seen from different perspectives. Reminder: flux is the product of number density and velocity, and quantum mechanical decay is always exponential.]
What would be the result of a non-relativistic calculation?

Problem 4

Derive the relativistic Doppler Effect: If a light-source that moves with (3-)velocity $\mathbf{u}$ emits photons with frequency $\omega_s = 2\pi\nu_s$, what is the frequency ω_o of these photons as seen by an observer that moves with a velocity $\mathbf{v}$?
Hint: Consider the 4-momentum of the photon $k^\mu = \left(\omega, \mathbf{k} = \frac{2\pi}{\lambda}\mathbf{n}\right)$, *where* $\mathbf{n}$ *is the unit vector pointing in the direction of propagation, and contract it with the 4-velocities of the source and observer.*

Problem 5

The (in)famous *twin paradox* considers the apparently symmetric situation of two astronaut twins. Alice stays at earth, while Bob heads away to a distant galaxy at some constant speed $v < c$, then turns around and heads back to earth with the same speed. This is exactly the same situation from Bob's perspective (Alice heading off, then turning back), it is then argued, so both 'expect' the other to be younger when they finally meet again.

a) Why is the situation actually *not* symmetric? Support your argument by drawing a space-time diagram that shows the world-lines of Alice and Bob, along with (equally spaced) lines of simultaneity as perceived by each of the two twins.

b) By which factor has Alice grown older more rapidly than Bob during the time they are separated?

c) Now let us assume that Bob decelerates with a constant rate g instead. Assuming small velocities, we can then transform from the 'rest frame' (at earth) to the accelerated frame by the coordinate transformation

$$\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} t \\ x - vt + \frac{1}{2}gt^2 \\ y \\ z \end{pmatrix}$$

Draw this situation in a space-time diagram, calculate the resulting proper time $d\tau$ for Bob (for whom we have $d\mathbf{x}' = 0$) and compare it to the proper time experienced by Alice. Relate the result to a) and b), and argue that this puts further evidence to the fact that the resolution of the 'paradox' is indeed a 'geometric' one, i.e. independent of the choice of coordinates or reference frame, and physically can be traced back to the immense acceleration that Bob undergoes when he turns back (in the original setup).