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http://www.uio.no/studier/emner/matnat/fys/FYS4160/v24/

Lecture spring 2024:

General Relativity

Problem sheet 2

⇝ These problems are scheduled for discussion on **Thursday, 8 February 2024**.

Problem 6

Some tensor algebra...

a) Consider a general rank (k,l) tensor

$$T = T^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_l}\ \hat{e}_{(\mu_1)} \otimes \ldots \otimes \hat{e}_{(\mu_k)} \otimes \hat{\theta}^{(\nu_1)} \otimes \ldots \otimes \hat{\theta}^{(\nu_l)}$$

- Show explicitly that $T\left(\hat{\theta}^{(\mu_1)}, \ldots, \hat{\theta}^{(\mu_k)}, \hat{e}_{(\nu_1)}, \ldots, \hat{e}_{(\mu_l)}\right) = T^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_l}$
- Use this result to evaluate the action of T on k arbitrary dual vectors $\omega_{(i)}$, $i = 1, \ldots, k$, and l arbitrary vectors $V^{(j)}$, $j = 1, \ldots, l$.

b) Consider a rank (0,2) tensor S and a rank $(2,0)$ tensor T.

- Write S and T as linear combinations of an arbitrary choice of basis vectors in the respective vector space. Now anti-symmetrize the *components* of S, and introduce $\tilde{S}$ as the tensor that has these components (when retaining the same arbitrary basis vectors that you used to decompose S). In the same way, construct a tensor $\tilde{T}$ from T.
- Show that $\tilde{T}(S) = T(\tilde{S}) = \tilde{T}(\tilde{S}) = \tilde{S}(\tilde{T})$. How would these coordinate-free statements look like in the index notation that you were familiar with from before?

Problem 7

Express the meaning of the various versions of the equivalence principles (EPs) in your own words, and discuss the differences. Based on this, point out at least one (thought) experiment that would test *i)* the weak EP (but not the other two), *ii)* the Einstein EP (but not the strong EP) *iii)* the strong EP (and argue why that is automatically a test of the others).

Problem 8

Both in flat space-time and in the presence of a gravitational field (but in the absence of any other forces), the equation of motion of a test particle in the coordinates ξ^μ of a freely falling system is given by the laws of Newton / special relativity, i.e. $\ddot{\xi}^\mu \equiv d^2\xi^\mu/d\tau^2 = 0$. Transforming to another set of coordinates x^μ, with $d\xi^\mu = (\partial\xi^\mu/\partial x^\nu)\, dx^\nu \equiv \xi^\mu_{,\nu} dx^\nu$, this equation becomes

$$\ddot{x}^\mu + \Gamma^\mu_{\rho\sigma}\dot{x}^\rho\dot{x}^\sigma = 0\,,$$

where the Christoffel symbols are given by

$$\Gamma^\mu_{\rho\sigma} = \frac{\partial x^\mu}{\partial\xi^\tau}\frac{\partial^2\xi^\tau}{\partial x^\rho\partial x^\sigma}\,.$$

In arbitrary coordinates (and even, as we will see, spacetimes), the line element takes the form $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. Using $g_{\mu\nu} = \eta_{\rho\sigma}\frac{\partial\xi^\rho}{\partial x^\mu}\frac{\partial\xi^\sigma}{\partial x^\nu}$ (why?), show that the above expression for $\Gamma^\mu_{\rho\sigma}$ is identical to the canonical form of the Christoffel symbols:

$$\Gamma^\mu_{\rho\sigma} = \frac{1}{2}g^{\mu\nu}\left(g_{\nu\rho,\sigma} + g_{\nu\sigma,\rho} - g_{\rho\sigma,\nu}\right)\,,$$

where $g_{\mu\nu}$ is the metric in the coordinate system x^μ and $g^{\mu\nu}$ is its inverse.