Torsten Bringmann (torsten.bringmann@fys.uio.no)
http://www.uio.no/studier/emner/matnat/fys/FYS4160/v24/

Lecture spring 2024:

General Relativity

Problem sheet 5

⇝ These problems are scheduled for discussion on **Thursday, 29 February 2024**.

Legend

* If pressed for time, make sure to try solving the other problem(s) before attempting this one

Problem 15*

The exterior derivative takes a p-form A and returns a $p+1$ form $\mathrm{d}A$, which in components is given by

$$(\mathrm{d}A)_{\mu_1 \ldots \mu_n} = (p+1)\, \partial_{[\mu_1} A_{\mu_2 \ldots \mu_{p+1}]}$$

Demonstrate the following properties of the exterior derivative:

a) $\mathrm{d}(\omega \wedge \eta) = (\mathrm{d}\omega) \wedge \eta + (-1)^p \omega \wedge (\mathrm{d}\eta)$, where ω is any p-form and η a q-form.

a) $\mathrm{d}(\mathrm{d}A) = 0$

Problem 16

Let f smoothly map a domain U of the $xy-$plane (in $\mathbb{R}^2$) into $\mathbb{R}^3$ by the formula $f(x,y) = (x, y, F(x,y))$ so that $M = f(U)$ is the surface $z = F(x,y)$. Given any point $p \in M$ describe the tangent space T_pM. Show that the area of the surface is given by

$$\int_U \sqrt{1 + \left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2}.$$

[Note that you can do the latter by simply starting with the Euclidian metric in 3D, and then substituting dx, dy and dz, as we have just done in the lecture for the simple case of a sphere. If you are interested in a much 'neater' and more general way of doing this, closer to the spirit of differential geometry, have a look at Appendix A in the book – and convince yourself that you get the same result.]

Problem 17

A unit sphere $\mathbb{S}^2$ can obviously be embedded in 3D space. Using the standard coordinates $x^\mu = (\theta, \phi)$ on the sphere, and cartesian coordinates $x'^\mu = (x, y, z)$ in 3D, the corresponding map is given by

$$x'^\mu(\theta, \phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)\,.$$

We have shown in the lecture (see also Appendix A in the book) that a Euclidian metric in 3D *induces* the standard metric on the sphere, $ds^2 = d\theta^2 + \sin^2\theta d\phi^2$. Let us now assume, instead, that the 3D metric takes the form

$$ds^2 = \frac{1}{1-z^2}dx^2 + \frac{1}{1-z^2}dy^2 + \frac{1-2z^2}{(1-z^2)^2}dz^2\,.$$

What is the induced metric on the object we defined earlier as $\mathbb{S}^2$ now (if we define it by the same map $x^\mu \to x'^\mu$ as above)? Discuss the result!

Problem 18

Let $T^2 = S^1 \times S^1$ be the torus, and let $\varphi : T^2 \longrightarrow \mathbb{R}^4$ be given by

$$\begin{aligned} x^1 \circ \varphi(\theta_1, \theta_2) &= \cos(\theta_1), \\ x^2 \circ \varphi(\theta_1, \theta_2) &= \sin(\theta_1), \\ x^3 \circ \varphi(\theta_1, \theta_2) &= 2\cos(\theta_2), \\ x^4 \circ \varphi(\theta_1, \theta_2) &= 2\sin(\theta_2), \end{aligned}$$

where $x^1, ..., x^4$ are the Euclidian coordinates on $\mathbb{R}^4$, and θ^1 and θ^2 are angular coordinates on T^2.

Express the Riemann metric induced on T^2 by φ (from the Euclidean metric $\mathbb{R}^4$) in terms of the coordinates θ^1 and θ^2 (that is, compute $g_{ij}(\theta^1, \theta^2)$)! What is the volume of T^2 relative to this metric?