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http://www.uio.no/studier/emner/matnat/fys/FYS4160/v24/

Lecture spring 2024:

General Relativity

Problem sheet 6

⇝ These problems are scheduled for discussion on **Thursday, 7 March 2024**.

Legend

* If pressed for time, make sure to try solving the other problem(s) before attempting this one

Problem 19

From your classes in 3D vector calculus, you are familiar with the expressions for the divergence ($\nabla \cdot \mathbf{A}$) and curl of a vector ($\nabla \times \mathbf{A}$) in spherical coordinates. Now re-derive these expressions by using the definition of the covariant derivative and the Christoffel symbols!
[Hint: first state these expressions in Euclidian orthonormal coordinates, in which they take a particularly simple form. Then determine the metric in spherical coordinates by an explicit coordinate transformation; use this to express the Christoffel symbols.]

Problem 20

The metric for a three-sphere can be written as

$$ds^2 = \mathrm{d}\psi^2 + \sin^2\psi \left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right)$$

a) Calculate the Christoffel symbols $\Gamma^\rho_{\mu\nu}$ by varying the length of a time-like curve $\tau = \int d\tau$.
[Hint: follow the derivation of the geodesic equation – but instead of using the Euler-Lagrange equations, directly vary the integrand (note that you can use the 'trick' of considering the square of the integrand, because $u^\mu u_\mu = -1$, which simplifies the calculation). In the end, you can simply read off the Christoffel symbols as the coefficients in front of the $u^\mu u^\nu$ terms.]

b) Calculate the Riemann tensor, the Ricci tensor and the Ricci scalar.

Problem 21*

Show that the vanishing of the Riemann tensor is a sufficient (and necessary) condition for the spacetime to be Minkowskian *everywhere* on the manifold, i.e. a (single, global) coordinate transformation will bring $g_{\mu\nu}$ into the form $\eta_{\mu\nu}$. Discuss: does this imply that spacetime is a globally trivial Minkowski space?

Problem 22*

Consider a vector $V \in T_{x_0}$ that is parallel-transported from some point x_0 on the manifold to a point $x_0 + \Delta x$, and from there to a point $y = x_0 + \Delta x + \delta x$. Here, both Δx and δx denote small displacements (for some coordinate choice x^μ).[1] Denote the resulting vector as $V_a \in T_y$. Now parallel-transport instead the vector V from x_0 $x_0 + \delta x$ and from there to y, and denote the resulting vector as $V_b \in T_y$. Show that

$$V_b - V_a = R^\mu{}_{\tau\sigma\nu} V^\sigma (\Delta x)^\nu (\delta x)^\tau \,.$$

Discuss the result.

[1]Technically, you could consider a curve $x^\mu(\lambda)$, where λ is not necessarily an affine parameter, and define the above-mentioned points e.g. as $x_0^\mu \equiv x^\mu(\lambda = 0)$, $(x_0 + \Delta x)^\mu \equiv x^\mu(\lambda = 1)$ and $(x_0 + \Delta x + \delta x)^\mu \equiv x^\mu(\lambda = 2)$. Then, $(\Delta x)^\mu = \delta\lambda(dx^\mu/d\lambda)$ for the first displacement etc.