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http://www.uio.no/studier/emner/matnat/fys/FYS4160/v24/

Lecture spring 2024:

General Relativity

Problem sheet 7

$\rightsquigarrow$ These problems are scheduled for discussion on **Thursday, 14 March 2024**.

Legend
* If pressed for time, make sure to try solving the other problem(s) before attempting this one

Problem 23

In Minkowski space the d'Alembertian operator is given by $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$, which readily generalizes to the curved space-time (and coordinate-invariant) version $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$. Show that this latter expression is equivalent to

$$\Box f = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\nabla^\mu f\right) ,$$

a form which can be advantageous for calculations in practice.

Problem 24

Consider the stress-energy tensor for a perfect fluid,

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + \eta^{\mu\nu} p$$

a) Use the conservation of this tensor, $\partial_\mu T^{\mu\nu} = 0$, in the non-relativistic limit to derive familiar results from classical hydrodynamics. In particular, derive the

- continuity equation, $\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0$
- Euler Equation, $\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{\nabla p}{\rho}$

[Hint: Have a look at 1.9 in the book. ;) But make sure that you really understand each step!]

b)* Now generalize the result to the Navier-Stokes equations, by allowing for off-diagonal (shear) terms in $T_{\mu\nu}$.

Problem 25

Noether's theorem states that for every *symmetry* in the Lagrangian – a field transformation that leaves the equations of motion invariant – there is a *conserved quantity.*

There thus exists a 4-vector j_N^μ that is divergence-free, $\partial_\mu j_N^\mu = 0$, and the conserved quantity (the 'Noether charge') is given by $Q_N = \int d^3x \, j_N^0$. You are encouraged to look up the general proof (it's one of the most important results in theoretical physics – and actually not that difficult!), which allows you to compute j_N^μ for any such field transformation.

Let us now consider the specific case of space-time translations $x^\nu \to x^\nu + a^\nu$, where a^μ is a constant 4-vector. Any theory formulated in Minkowski space must be invariant under these transformations, so there must be *four* conserved currents $j_N^{(\nu)\,\mu} \equiv T^{\mu\nu}$ (one for each value of ν) and the corresponding charges are the total energy and 3-momentum of the system, respectively. The energy-momentum tensor, of any field $\phi^a(x)$, obtained in this way is given by

$$T^{\mu\nu} = -\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)} \partial^\nu \phi^a + \eta^{\mu\nu} \mathcal{L} \,,$$

where $\mathcal{L} = \mathcal{L}(\phi^a, \partial_\mu \phi^a)$ is the Lagrangian (density). Now consider the general Lagrangian of a single scalar field in an arbitrary potential, $\mathcal{L} = -\frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2$, and derive the stress-energy tensor! Try to give an interpretation of the individual terms that appear in the resulting expression.