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http://www.uio.no/studier/emner/matnat/fys/FYS4160/v24/

Lecture spring 2024:

General Relativity

Problem sheet 14

⇝ These problems are scheduled for discussion on **Thursday, 16 May 2024**.

Problem 41

Consider the line element of the FRLW metric

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a^2(t)\left[\frac{dr^2}{1-\kappa r^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right]$$

and the stress-energy tensor of a perfect fluid, which in the rest frame of the fluid is given by

$$T^{\mu\nu} = \mathrm{diag}(\rho, p, p, p)\,.$$

Compute the $(\mu, \nu) = 00$- and 11-component of Einstein's field equations,

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}\mathcal{R}\, g_{\mu\nu} = 8\pi G T_{\mu\nu} - \Lambda\, g_{\mu\nu}\,.$$

Hint (for a much *faster calculation): Convince yourself that $\Gamma^\mu_{00} = 0$ and that the relevant components of the Riemann tensor are given by ${\mathcal{R}^i}_{0j0}$, ${\mathcal{R}^0}_{i0j}$ and ${\mathcal{R}^k}_{ikj}$. Write the latter as ${\mathcal{R}^k}_{ikj} = {}^{(3)}\mathcal{R}_{ij} + ...$ and use the fact that the 3D Ricci tensor takes a very simple form for a space of constant curvature, namely ${}^{(3)}\mathcal{R}_{ij} = 2k\gamma_{ij}$, where γ_{ij} is the metric on the spatial, maximally symmetric three-manifold (see lecture).*

Problem 42

In this problem, we consider the lookback time and the age of the universe:

a) Show that the Hubble rate H as a function of the redshift $z \equiv a^{-1} - 1$ is given by

$$H^2 = H_0^2\left[\Omega_r(1+z)^4 + \Omega_\mathrm{m}(1+z)^3 + \Omega_k(1+z)^2 + \Omega_\Lambda\right]\,,$$

where H_0 is the present value of H and Ω_i are the present ratios of the energy density ρ_i of some component i to the critical density ρ_c (and $\Omega_k \equiv -\kappa/\dot{a}^2$).

b) Use this result to express the 'lookback time', $t_0 - t_1$, as a redshift integral from today ($z = 0$) to the redshift z_1 at time t_1.

c) Derive (analytically) the age of the universe for the so-called Einstein de Sitter universe ($\sum_i \Omega_i = \Omega_\mathrm{m} = 1$). What is the numerical value for $H_0 \equiv 100\,h\,\mathrm{kms}^{-1}\mathrm{Mpc}^{-1}$, where $h = 0.674 \pm 0.005$ according to the most recent measurements? Discuss the physical significance of the 'age' of the universe!

d) Determine numerically the age of the universe for the current best-fit values of the cosmological parameters $\Omega_\mathrm{m} = 0.31$ and $\Omega_\Lambda = 0.69$ (assume that $\Omega_m \gg \Omega_r$). Which impact does a $\sim 2\%$ error on these quantities have for the result? For the radiation density today, we have $\Omega_r \sim 10^{-4}\,\Omega_\mathrm{m}$; how much does such a contribution affect the total age of the universe?

Problem 43

This problem considers applications of the cosmological particle horizon.

a) Discuss the concept of the 'size of the universe' (i.e. what is meant by this?)!

b) What is the size of the visible universe today (first state the exact expression, then evaluate it numerically in Gpc)?

c) What was it at the time of matter-radiation equality ($z_\mathrm{eq} \sim 3500$), what at the time when the CMB photons were emitted ($z_\mathrm{rec} \sim 1300$)?

d) What is thus the maximal angular size of a region on the sky (today) that was causally connected at the time when the CMB photons were released?