UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Exam in: General Relativity (FYS4160)

Day of exam: June 10, 2024

Exam hours: 4 hours

This examination paper consists of 4 pages. (including the title page)

Appendices: none

Permitted materials: 3 A4 pages (two-sided) with own notes.

Make sure that your copy of this examination paper is complete before answering.

Final exam

Lecture spring 2024: General Relativity (FYS4160)

$\rightsquigarrow$ *Carefully **read all questions** before you start to answer them! Note that you do not have to answer the problems in the order presented here, so try to answer those first that you feel most sure about. In particular, questions marked with an* **asterisk** *($*$) require slightly more heavy calculations / algebra – so once you feel you are getting stuck make sure to move on (and get back to these later).*

Also note that you can (largely) solve each of the subproblems even if you haven't managed to solve the previous subproblems – but this requires to use the information explicitly stated in the previous subproblems. Keep your descriptions as short and concise as possible! Answers given in English are preferred, but feel free to write in Norwegian if you struggle with formulations! Maximal number of available points: **44**.

Good luck !!!

Problem 1 *(6 points)*

a) You hear someone explain that 'in flat spacetime (and hence in special relativity) the Christoffel symbols $\Gamma^{\mu}_{\rho\sigma}$ vanish, but in curved spacetime (and hence in general relativity) they do *not* vanish.' Is that right or wrong (or something in between)? Correct the statement as necessary! *(3 points)*

b) Consider a light pulse emitted horizontally in vacuum near the Earth's surface. Use the equivalence principle to calculate the vertical distance the light has fallen after travelling 1 km! One digit in precision is sufficient for the answer. *(3 points)*

Problem 2 *(9 points)*
In this problem we consider the Schwarzschild metric in spherical coordinates,

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) .$$

a) Briefly show that $E \equiv (1 - 2GM/r)\,\dot{t}$ and $L \equiv r^2\dot{\phi}\sin^2\theta$, with $\cdot \equiv d/d\tau$, are conserved quantities for observers in free fall. *(3 points)* *b) From now on, we restrict ourselves to motions in the equatorial plane ($\theta = \pi/2$). In the lecture, we then used the conservation of E and L to derive the following equation for a time-like observer in free-fall:

$$\frac{1}{2}\left(E^2 - 1\right) = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 - \frac{GM}{r} + \frac{L^2}{2r^2} - \frac{GML^2}{r^3} \equiv \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{\text{eff}}(r)\,.$$

Use this expression to determine L^2 and E^2 for a stable circular orbit at $r = r_c$. For the next problem you'll need the combination $L/E = \alpha\sqrt{GMr_c^3}/(r_c - 2GM)$, where α is a constant. *(3 points)*

c) Now calculate the orbital period for such an orbit as measured by a far-way observer. Using the result from b), simplify the expression by expressing it in terms of r_c and M only. How does this compare to the Newtonian result? *(3 points)*

Problem 3 *(14 points)*
When studying gravitational waves, we have sometimes adopted the so-called *traceless-transverse* (TT) gauge.

a) State the defining conditions on the metric perturbations in this gauge. Briefly describe how we convinced ourselves that this is a physically possible gauge fixing condition. Did this involve any assumptions on the matter part of Einstein's equations, i.e. the form of $T^{\mu\nu}$? *(4 points)*

b) Demonstrate that observers initially at rest remain at rest in this gauge, i.e. their coordinate positions do not change as a gravitational wave passes through. It is sufficient to show this at linear order in $h_{\mu\nu}$! *(4 points)*

c) As we derived in the lecture, solutions to the wave equation in this gauge are of the form $h_{\mu\nu} = C_{\mu\nu} \exp[ik_\sigma x^\sigma]$. State the conditions that the wave vector k_σ and the polarization tensor $C_{\mu\nu}$ have to satisfy. For each of the possible polarizations of the gravitational wave passing through, identify the directions where spatial *distances* remain constant in time! *(6 points)*

Problem 4 *(15 points)*
Let us consider the flat FRLW metric which, in Cartesian coordinates, is given by

$$ds^2 = -dt^2 + a^2(t)d\mathbf{x}^2\,,$$

with a being the cosmological scale factor.

a) Perform a change of coordinates to conformal time η, with $d\eta/dt \equiv a^{-1}$. Discuss the form of the light cones in these new coordinates; what is the main change w.r.t. the original coordinates? *(3 points)*

*b) Compute the spatial part of the geodesic equation, by directly varying the line element for a finite distance $\Delta s = \int ds$ along some curve $x^\mu(\lambda)$. You should find an expression of the form $d^2x^i/d\tau^2 = A\dot{x}^i(a'/a)\dot{\eta}$, where A is a constant, $\cdot \equiv d/d\tau$ and $' \equiv d/d\eta$. *(5 points)*
[Hint: Changing $\lambda \to \tau$ sufficiently early greatly simplifies the calculation.]

c) Consider $\mathbf{k} \equiv a\dot{\mathbf{x}}$. What is the physical significance of this quantity? Use the result from the previous problem to determine how $\mathbf{k}$ scales with the scale-factor a for an observer in free-fall (i.e. determine the 'n' in $\mathbf{k} \propto a^n$). Interpret the result. *(4 points)*

d) Show that if a relativistic particle species decouples from a thermal bath at a given time t_1, with a thermal equilibrium distribution, it will maintain this distribution with a temperature which is simply redshifted by the expansion of the Universe. What is the general form of a spectral distribution $f(\mathbf{p})$ for which this is the case ? *(3 points)*

Useful formulae

$$c \approx 3 \cdot 10^{10}\,\mathrm{cm/s} \quad (1)$$

$$\Gamma^\mu_{\rho\sigma} = \frac{1}{2} g^{\mu\nu}\left(g_{\rho\nu,\sigma} + g_{\nu\sigma,\rho} - g_{\rho\sigma,\nu}\right) \quad (2)$$

$$f^{\mathrm{eq}}(p) = \frac{1}{e^{E/T} \pm 1} \quad (3)$$