Solutions [and grading guidelines] to final exam in FYS4160

[General grading guidelines: Stated points are given for arriving at the respective expression in a fully satisfactory way. Whole point subtractions for bad logic, physically wrong statements or expressions that are *physically* wrong (e.g. wrong dimensions)! 'Obvious' small math errors (e.g. wrong prefactors) only 0.5 pt. When possible, no subtraction for follow-up mistakes. Up to 1 extra point for outstanding explanations or demonstrating special insight (but this cannot result in more points than available for a given problem).]

Problem 1

a) In flat spacetime there *exist* coordinates (Cartesian coordinates, in particular), where $\Gamma^\mu_{\rho\sigma} = 0$ *everywhere* **[1 pt]**. In general, e.g. in spherical coordinates, this is *not* the case. If spacetime is curved, one can alway find coordinates such that $\Gamma^\mu_{\rho\sigma} = 0$ *in any given point* **[1 pt]** – but it is impossible to find a coordinate system such that $\Gamma^\mu_{\rho\sigma} = 0$ everywhere **[1 pt]**.

b) According to the equivalence principle, a freely falling observer should locally not see any effect of gravity and hence observe the light pulse propagate along a straight line **[1 pt]**. The time it takes for light to traverse 1 km is given by $\Delta t = L/c = 1/(3 \cdot 10^5)\,\mathrm{s} \approx 3 \cdot 10^{-6}\,\mathrm{s}$. With respect to the free-fall frame, the ground 'moves upwards' with a constant acceleration of $g \approx 9.8\mathrm{m/s}^2$ **[1 pt]**. In other words, after Δt the photons have fallen a vertical distance of

$$\Delta h = \frac{1}{2} g (\Delta t)^2 \approx 5 \cdot 10^{-11}\,\mathrm{m}\,.\textbf{[1 pt]} \tag{1}$$

(Note that in principle we should have integrated $d^2z/d\tau^2 = g$ here – but since $\tau \simeq t$ for the small velocities that we consider, we could directly take the Newtonian result. **[1 bonus point for discussing this]**)

Problem 2

a) The spacetime has two manifest isometries, i.e. coordinates that the metric does not depend on, namely t and ϕ. The corresponding Killing vectors are given by $K^\mu \equiv (\partial_t)^\mu = (1,0,0,0)$ and $R^\mu \equiv (\partial_\phi)^\mu = (0,0,0,1)$ **[1 pt]**. Contracting a Killing vector with the momentum $p^\mu = m\, dx^\mu/d\tau$ gives a quantity that is conserved along geodesics. Hence,

$$E \equiv -K^\mu \frac{dx_\mu}{d\tau} = -g_{\mu\nu} K^\mu \frac{dx^\nu}{d\tau} = -g_{tt} \frac{dt}{d\tau} = \left(1 - \frac{2GM}{r}\right) \dot{t} \tag{2}$$

$$L \equiv R^\mu \frac{dx_\mu}{d\tau} = g_{\mu\nu} R^\mu \frac{dx^\nu}{d\tau} = g_{\phi\phi} \frac{d\phi}{d\tau} = r^2 \dot{\phi} \sin^2\theta \tag{3}$$

are conserved in free fall **[1 pt for each of these]**.

b) The stable circular orbit occurs at the miminum of V_{eff}, i.e.

$$0 = V'_{\text{eff}} = \frac{GM}{r_c^2} - \frac{L^2}{r_c^3} + \frac{3GML^2}{r_c^4} = GMr_c^2 - L^2 r_c + 3GML^2\,, \tag{4}$$

and hence [1 pt]

$$L^2 = \frac{GMr_c^2}{r_c - 3GM}\,. \tag{5}$$

The energy is obtained by inserting this value of L^2 in the expression stated in the problem (for $dr/d\tau = 0$):

$$\frac{1}{2}\left(E^2 - 1\right) = -\frac{GM}{r_c} + \left(\frac{1}{2r_c^2} - \frac{GM}{r_c^3}\right)\frac{GMr_c^2}{r_c - 3GM}\text{[1 pt]} \tag{6}$$

$$= -\frac{GM}{r_c} + \frac{1}{2}\frac{GM(1 - 2GM/r_c)}{r_c - 3GM} \tag{7}$$

Solving for E^2:

$$E^2 = 1 - \frac{2GM}{r_c} + \frac{GM(1-2GM/r_c)}{r_c - 3GM} \tag{8}$$

$$= \frac{r_c(r_c - 3GM)}{r_c(r_c - 3GM)} - \frac{2GM(r_c - 3GM)}{r_c(r_c - 3GM)} + \frac{GM(r_c - 2GM)}{r_c(r_c - 3GM)} \tag{9}$$

$$= \frac{(r_c - 2GM)^2}{r_c(r_c - 3GM)}.\text{[1 pt]} \tag{10}$$

Hence, $L/E = \sqrt{GMr_c^3}/(r_c - 2GM)$.

c) The orbital frequency as measured by a far-away observer is

$$\Omega = \frac{d\phi}{dt} \overset{\text{[1 pt]}}{=} \frac{\dot{\phi}}{\dot{t}} = \frac{L/r_c^2}{E/(1 - 2GM/r_c)} = \frac{L}{E}\frac{r_c - 2GM}{r_c^3} = \alpha\sqrt{\frac{GM}{r_c^3}}\,,\text{[0.5 pt]} \tag{11}$$

where in the last step we used the information given in problem 2b) [though we just calculated $\alpha = 1$]. From this, we can trivially get the orbital period as $T = 2\pi/\Omega$ **[0.5 pt]**.

In the Newtonian case, we can equate gravitational and centrifugal acceleration to obtain

$$\frac{GM}{r_c^2} = \Omega^2 r_c \quad \rightsquigarrow \quad \Omega = \sqrt{\frac{GM}{r_c^3}}\,, \tag{12}$$

i.e. exactly the same expresssion [1 pt]. This exact agreement is somewhat surprising and in fact completely coincidental. Actually, the agreement is *not* exact as the interpretation of r is different – the closer to the Horizon, the more differs r from the 'standard' radial variable in Euclidian space **[1 bonus point for discussing this aspect]**.

Problem 3

a) Writing the metric as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,, \tag{13}$$

the TT gauge-fixing conditions are given by **[1 pt]**

$$h_{\mu 0} = 0 \qquad \partial^i h_{ij} = 0\,, \qquad \eta^{ij} h_{ij} = 0\,. \tag{14}$$

To understand that this gauge is possible, we first noticed that the linearized Einstein equations are invariant under $h_{\mu\nu} \to h_{\mu\nu} + 2\partial_{(\mu}\xi_{\nu)}$ for arbitrary functions ξ^μ. By choosing these four functions appropriately, it is then always possible to apply the transverse gauge fixing condition, $\partial^i h_{0i} = 0$ and $\partial^i s_{ij} = 0$, where s_{ij} is the traceless part of h_{ij} (and contains the only propagating degrees of freedom of Einstein's equations). The constraint equations for the other metric perturbation components then evaluated to **[up to 2 pts for any reasonable description of these steps]** $w_i \equiv h_{0i} = 0$, $\phi \equiv -h_{00}/2\phi = 0$ and $\psi \equiv -\frac{1}{6}\eta^{ij}h_{ij} = 0$ *in vacuum* **[1 pt]**. The latter is important – for example, one cannot use the TT gauge to describe the *production* of GWs.

b) In order to answer this question, we need to consider the geodesic equation **[1 pt]**,

$$\frac{d^2x^i}{d\tau^2} = -\Gamma^i_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\,. \tag{15}$$

The Christoffel symbols for a metric of the form in Eq. (13) are given by

$$\Gamma^\mu_{\rho\sigma} = \frac{1}{2}g^{\mu\nu}\left(g_{\rho\nu,\sigma} + g_{\nu\sigma,\rho} - g_{\rho\sigma,\nu}\right) \tag{16}$$

$$= \frac{1}{2}\eta^{\mu\nu}\left(h_{\rho\nu,\sigma} + h_{\nu\sigma,\rho} - h_{\rho\sigma,\nu}\right) + \mathcal{O}(h^2)\,.\textbf{[1 pt]} \tag{17}$$

For observers initially at rest, we have $u^\mu \equiv dx^\mu/d\tau = (1,\mathbf{0}) + \mathcal{O}(h)$. Hence, the r.h.s. becomes to leading order

$$-\Gamma^i_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -\Gamma^i_{00}u^0u^0 = -\Gamma^i_{00} = \frac{1}{2}\eta^{i\nu}\left(2h_{0\nu,0} - h_{00,\nu}\right)\,,\textbf{[1 pt]} \tag{18}$$

which vanishes identically in TT gauge. Hence, $\frac{d^2x^i}{d\tau^2} = 0 \rightsquigarrow x^i = const.$.**[1 pt]**

c) The general solution for the wave equation in TT gauge is

$$h_{\mu\nu} = C_{\mu\nu}e^{ik_\sigma x^\sigma}\,, \tag{19}$$

where k^μ is the (light-like) wave vector – thus satisfying $k^2 = 0$ **[1 pt]**. The conditions on the polarization tensor follow from the TT conditions stated in 3a). For a plane GW propagating in x^3 direction, e.g., we have $k^\mu = (\omega, 0, 0, \omega)$ and **[1 pt, but only if noting that this depends on the choice of z-axis.]**

$$C_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{20}$$

In order to measure distances, we need to consider the spacetime interval

$$ds^2 = (\eta_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu \tag{21}$$

$$= -dt^2 + d\mathbf{x}^2 + h_+(dx^2 - dy^2)e^{i\omega(-t+z)} + 2h_\times(dx\,dy)e^{i\omega(-t+z)}\,,\textbf{[1 pt]} \tag{22}$$

where we have adopted the same *choice* for the z-axis, namely to align it with the direction of $\mathbf{k}$ (and it is understood that we only take the real part of the above expression).

The fact that the metric components $g_{\mu 3}$ are independent of time means, as also seen in the lecture, that there is no change in distance in the direction of the z-axis. Thus, we only need to consider directions in the $x-y$ plane **[1 pt]**. This means that, as we move along some space-like parameterized curve $(x(\lambda), y(\lambda))$ to calculate the distance, $\int d\lambda \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}$ with $\dot{x}^\mu \equiv dx^\mu/d\lambda$, we want to make sure that we do not pick up any time-dependence. For the '+' polarization ($h_\times = 0$), this implies

$$dx^2 - dy^2 = 0 \quad \rightsquigarrow \quad y = \pm x + const.\,, \textbf{[1 pt]} \tag{23}$$

while for the '×' polarization ($h_+ = 0$), we need

$$dx\, dy = 0 \quad \rightsquigarrow \quad y = const. \text{ or } x = const.\,. \textbf{[1 pt]} \tag{24}$$

In both cases, this describes directions at a 45° angle with respect to the polarization directions.

Problem 4

a) With $dt = a d\eta$, the metric now takes the form $ds^2 = -a^2(\eta)(d\eta^2 - d\mathbf{x}^2)$ **[1 pt]**. For light rays, $ds^2 = 0$, we have $d\eta^2 = d\mathbf{x}^2$, i.e. $d|\mathbf{x}|/d\eta = \pm 1$ **[1 pt]**. The light cones are thus always at 45° in these coordinate, independent of time; in the original coordinates, we instead have $d|\mathbf{x}|/dt = \pm a^{-1}(t)$, i.e. light cones that change with the expansion of the universe **[1 pt]**.

b) The distance along the curve $x^\mu(\lambda)$ is given by

$$\Delta s = \int ds = \int \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} d\lambda = \int a(\eta)\sqrt{\dot{\eta}^2 - \dot{\mathbf{x}}^2} d\lambda\,, \textbf{[1 pt]} \tag{25}$$

where $\cdot \equiv d/d\lambda$. Since we are only interested in the spatial part of the geodesic, we also only need to perform the variation with respect to $\dot{\mathbf{x}}$ (and $\mathbf{x}$, which the above expression however does not explicitly depend on):

$$0 \stackrel{!}{=} \delta_{\mathbf{x},\dot{\mathbf{x}}} \Delta s = \int a(\eta) \frac{d\left(\sqrt{\dot{\eta}^2 - \dot{\mathbf{x}}^2}\right)}{d\dot{x}^j} \delta(\dot{x}^j) d\lambda = \int a(\eta) \frac{-\dot{x}_j}{\sqrt{\dot{\eta}^2 - \dot{\mathbf{x}}^2}} \underbrace{\delta(\dot{x}^j)}_{\frac{d}{d\lambda}\delta x^j} d\lambda\,. \textbf{[1 pt]} \tag{26}$$

Integrating by parts **[1 pt]**, we see that this equation can only be satisfied (for arbitrary δx^j) if

$$\frac{d}{d\lambda}\left(a(\eta)\frac{\dot{x}_j}{\sqrt{\dot{\eta}^2 - \dot{\mathbf{x}}^2}}\right) = 0\,. \tag{27}$$

From now on, we take $\lambda \to \tau$ as this is required to bring the geodesic equation into its standard form. Then, we can use $d\tau^2 = -g_{\mu\nu}dx^\mu dx^\nu$ to conclude that $1 = a\sqrt{\dot{\eta}^2 - \dot{\mathbf{x}}^2} = const.$ **[1 pt]**. This simplifies the above condition to

$$0 = \frac{d}{d\tau}\left(a^2 \dot{x}_j\right) = 2a\dot{a}\dot{x}_j + a^2 \ddot{x}_j\,, \tag{28}$$

In other words, we find $A = -2$: **[1 pt]**

$$\frac{d^2x^i}{d\tau^2} = -2a^{-1}\dot{x}^i a'\dot{\eta} = -2a^{-1}\dot{x}^i a'\sqrt{a^{-2} + \dot{\mathbf{x}}^2} \tag{29}$$

c) Comoving / coordinate distances are described by x^i, so the 'comoving' momentum is defined w.r.t. motion in this coordinate system as $p^i \equiv m\, dx^i/d\tau$. Multiplying this by the scalar factor makes $\mathbf{k}$ the 'physical' momentum, per unit mass **[1 pt]**. We can then use

$$\frac{d}{d\tau}k^i \equiv \frac{d}{d\tau}(a\dot{x}^i) = \dot{a}\dot{x}^i + a\ddot{x}^i = \dot{a}\dot{x}^i + A\dot{a}\dot{x}^i\,, \text{[1 pt]} \tag{30}$$

where in the last step we used the result stated in the problem formulation, noting that $a'\dot{\eta} = \dot{a}$. We thus have

$$\frac{d}{d\tau}k^i = (A+1)\frac{\dot{a}}{a}k^i \quad \rightsquigarrow \quad k^i \propto a^{A+1}\,.\text{[1 pt]} \tag{31}$$

Using $A = -2$ from the previous result, this just means that physical momenta of freely falling observers redshift a $1/a$ – which is the expected result **[1 pt]**.

d) The momentum $p \equiv |\mathbf{p}|$ in the thermal equilibrium distributions stated among the 'useful formula', through $E = \sqrt{p^2 + m^2}$, is the *physical* momentum **[1 pt for realizing this in some form]**. In other words, we have $\mathbf{p} = m\mathbf{k} \propto a^{-1}$ for *free* particles (after decoupling) from the previous problem. At the point of decoupling, $t = t_1$, we still have a thermal distribution of relativistic particles (for which $E = p$):

$$f(p)|_{t_1} = \frac{1}{\exp(p_1/T_1) \pm 1}\,, \tag{32}$$

where $p_1 \equiv p(t_1)$ and $T_1 = T(t_1)$. For later times, each of the momenta redshifts, leading to a distribution

$$f(p) = \frac{1}{\exp(p_1/T_1) \pm 1} = \frac{1}{\exp[p(a/a_1)/T_1] \pm 1} \equiv \frac{1}{\exp(p/T_{\rm dec}) \pm 1}\,, \tag{33}$$

that indeed takes the same form as the initial one if we define the 'temperature' of the decoupled species as $T_{\rm dec} \equiv T_1(a_1/a)$ **[1 pt]**. Note that we did not actually use that f describes a *thermal* distribution; the same derivation would work for any distribution of the form $f(p, T) = f(p/T)$ **[1 pt]**. Notably, this is the case for an initially thermal distribution both in the highly relativistic and in the highly non-relativistic limit – but not in general.