FYS4160 - General Relativity
Problem Set 0 Solutions
Spring 2024

These solutions are credited to Jake Gordin, who wrote them in the years 2020-23.

If you spot any typos, mistakes, don't hesitate to contact me at halvor.melkild@fys.uio.no. For any physics related question please use the forum at astro-discourse.uio.no.

The idea of these solutions is to give you a sense of what a 'model' answer should be, and they also elaborate on some discussions from the help sessions. I try to make them "pedagogical": i.e. hopefully comprehensive and most steps should be explained.

Problem 1. Index notation

a)
- $(\nabla S)_i = \partial_i S$
- $\nabla \cdot \mathbf{A} = \partial_i A^i$
- $(\nabla \times \mathbf{A})_i = \epsilon_{ijk}\partial^j A^k$
- $\mathrm{Tr}(M) = M_{ij}\delta^{ij} = M^i{}_i$
- $(M_{ij})^T = M_{ji}$

b)
- $\nabla \cdot (\nabla \times \mathbf{A}) = 0$, where $\mathbf{A}$ is a 3D vector:

  $$\partial_i(\epsilon^{ijk}\partial_j A_k) = \epsilon^{ijk}\partial_i\partial_j A_k = \epsilon^{ijk}\partial_{(i}\partial_{j)}A_k = \epsilon^{(ij)k}\partial_{(i}\partial_{j)}A_k = 0.$$

  Step three to four follows from equation (1.83) in Carroll. The answer is then zero because of the antisymmetry of ϵ^{ijk} – it contributes no symmetric part when summed over.

- $\nabla \times (\nabla S) = 0$, where S is a scalar:

  $\epsilon^{ijk}\partial_j(\partial_k S) = \epsilon^{ijk}\partial_{(j}\partial_{k)}S = \epsilon^{(ij)k}\partial_{(j}\partial_{k)}S = 0$ (Same argument as above.)

Problem 2. Index notation part 2, electric boogaloo

a)
- $\partial_\mu x^\nu = \delta^\nu_\mu$
  Right: $\partial_\mu x^\nu = \frac{\partial x^\nu}{\partial x^\mu} = \delta^\nu_\mu$
- $\partial_\mu x^\mu = 1$
  Wrong: $\partial_\mu x^\mu = \delta^\mu_\mu = 4$
- $\partial^\mu x^\nu = g^{\mu\nu}$
  Right: $\partial^\mu x^\nu = g^{\mu\rho}\partial_\rho x^\nu = g^{\mu\rho}\delta^\nu_\rho = g^{\mu\nu}$
- $T_\alpha{}^\beta{}_\gamma = g^{\beta\mu}T_{\alpha\mu\gamma} = g^{\mu\beta}T_{\alpha\mu\gamma}$
  Right: the metric is sym-metric (I'll show myself out).
- $T_\alpha{}^\beta{}_\beta = g_{\alpha\mu}g^{\beta\alpha}T^\mu{}_{\alpha\beta}$
  Wrong: this is a case of nonsensical notation, there are three α's - the sum over them is meaningless. Replace with a different dummy index to get $T_\alpha{}^\beta{}_\beta = g_{\alpha\mu}g^{\beta\nu}T^\mu{}_{\nu\beta} = g^{\beta\nu}T_{\alpha\nu\beta}$

b)
- All independent Lorentz scalars from two four-vectors A and B:

$A^\mu A_\mu,\ B^\mu B_\mu,\ A^\mu B_\mu$ (Lorentz scalar here simply means no free indices.)

- All independent Lorentz scalars from a rank-2 tensor T:

  $T^{\mu\nu}T_{\mu\nu},\ T^{\mu\nu}T_{\nu\mu},\ T^\mu{}_\mu,\ T^\mu{}_\nu T^\nu{}_\rho T^\rho{}_\mu,\ T^\mu{}_\nu T^\nu{}_\rho T^\rho{}_\lambda T^\lambda{}_\mu,\ \ldots$ (again, no free indices)

  The list can go on forever, but the list of <u>independent</u> scalars is still finite, as the number of components of T is finite.

- All independent Lorentz scalars from a rank-2 tensor T and two four-vectors A and B:

  $T^{\mu\nu}A_\mu A_\nu,\ T^{\mu\nu}A_\mu B_\nu,\ T^{\mu\nu}B_\mu A_\nu,\ T^{\mu\nu}B_\mu B_\nu,\ T^\mu{}_\mu A^\nu B_\nu,\ T^\mu{}_\nu T^\nu{}_\rho A^\rho A_\mu,\ T^{\mu\nu}T_{\mu\nu}A^\rho A_\rho,\ \ldots$ (no free indices again - double check though!)

- All independent Lorentz four-vectors from a scalar S and two four-vectors A and B:

  $SA^\mu,\ SB^\mu$ (Four-vector means one free index, so these are the only combinations.)