FYS4160 - General Relativity
Problem Set 1 Solutions
Spring 2024

These solutions are credited to Jake Gordin, who wrote them in the years 2020-23.

If you spot any typos, mistakes, don't hesitate to contact me at halvor.melkild@fys.uio.no. For any physics related question please use the forum at astro-discourse.uio.no.

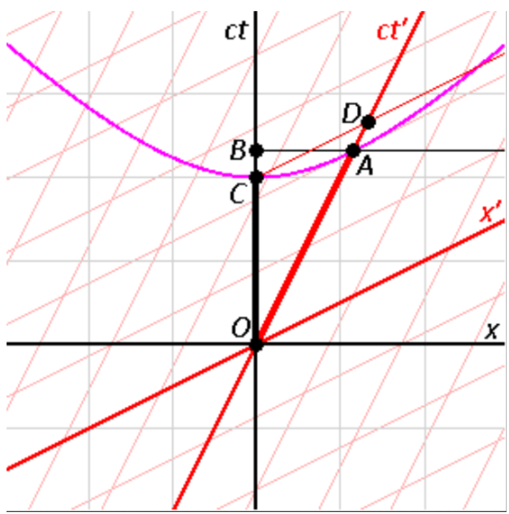
The idea of these solutions is to give you a sense of what a 'model' answer should be, and they also elaborate on some discussions from the help sessions. I try to make them "pedagogical": i.e. hopefully comprehensive and most steps should be explained.

Problem 3. Time dilation and the little muon that could

a) *i)* Using a spacetime diagram:

Starting from the origin O, we are in the clock's reference frame. The clock moves along the t' axis, while the observer continues along the t axis (remember we normally set $c = 1$, despite the diagram label).
The invariance of the spacetime interval implies time dilation, i.e. $\Delta t' \equiv |OA| < |OB| \equiv \Delta t$. Using from Pythagoras from the diagram and the invariance of Δs^2,

$$\begin{aligned}\Delta s^2 &= -\Delta t^2 + \Delta x^2 = (v^2 - 1)\Delta t^2 \\ &= -\Delta t'^2 + 0 \\ \implies \Delta t &= \gamma \Delta t' \text{ with } \gamma = (1 - v^2)^{-1/2}\end{aligned}$$



Time dilation. Note how hyperbolas $-t^2 + x^2 = -t'^2 + x'^2 = \pm k^2$ (with k a constant) connect points with equal spacetime intervals. Here, $|OA| = |OC| < |OB|$.

Image from Wikipedia, naturally. Here is a more thorough discussion of 'calibration hyperbolas', and here is another nice illustration.

ii) Using the explicit form of a Lorentz boost is some matrix multiplication:

$$\begin{pmatrix} t'_A \\ x'_A \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} t_A \\ x_A \end{pmatrix} \overset{x_A = vt_A}{\Longrightarrow} \Delta t' = t'_A = \gamma t_A - \gamma\beta^2 t_A = \gamma(1 - \beta^2)\Delta t = \Delta t/\gamma$$

b) This question dumps a lot of information at once, but we're actually asked for just two things: the velocity of the muons and their flux ratios (in both frames).

We are given the energy and mass of muons, so we can find their velocity using the full version of $E = mc^2$, namely $E^2 = p^2 + m^2$ with $p = \gamma m v$. This leads to

$$v = \sqrt{1 - \frac{m^2}{E^2}} = 0.994c \quad \text{and} \quad \gamma = 9.379\,.$$

Quite fast,[1] almost the speed of light. To find the flux ratio, imagine a box with volume V and N muons moving at velocity v through it. Flux is the number of things per m^2 per s, so

$$\phi = \frac{vN}{V}$$

which gives the flux ratio at the height h to the ground as $\phi_0/\phi_h = N_0/N_h$ (even in the Earth frame, where one of the box's dimensions is seen to be contracted, the γ factor cancels out in the ratio.) The number of muons is of course not fixed, and decays via $N(t) = N(0)\exp(-t/\tau)$. We're given the lifetime, so all that remains is to work out reference frames.

	Non-relativistic view	**Muon frame**	**Earth frame**
Distance	$L_0 = 10$ km	$L = L_0/\gamma = 1.066$ km	$L_0 = 10$ km
Travel time	$T = L_0/v = 33.5\ \mu$s	$T_0 = L/v = 3.57\ \mu$s	$T = L_0/v = 33.5\ \mu$s
Lifetime	$\tau_0 = 2.2\ \mu$s	$\tau_0 = 2.2\ \mu$s	$\tau = \gamma\tau_0 = 20.6\ \mu$s
N_0/N_h	$\exp(-T/\tau_0) = 2.44 \times 10^{-7}$	$\exp(-T_0/\tau_0) = 0.197$	$\exp(-T/\tau) = 0.197$
ϕ_0/ϕ_h	2.44×10^{-7}	0.197	0.197

In the muon rest frame, the distance to Earth is contracted. From the point of view of an observer on Earth, the clock of the moving muons is ticking slower. Yet the end result of the relativistic calculations of the flux ratio in both inertial frames is the same, as it should be. Why should it be so? Because the muons still make it to Earth all the same! Time and length may be relative, but the bare fact of how many particles make it to Earth is not.[2]

The non-relativistic calculation gives a completely different answer, also to be expected. The number is far lower – this makes sense, since without time slowing or length contracting there is no way any muon on average makes it to Earth before decaying.

There is a very nice video on the basic concept behind a muon's journey to Earth, should anyone be interested: https://www.youtube.com/watch?v=kGsbBw1IORg&ab_channel=SixtySymbols.

Problem 4. Relativistic Doppler effect

We define three frames for this problem. We call the frame S the frame of an external observer. In this frame the velocity of the external observer is zero, and our light source and receiver are seen in motion with respective 4-velocities $u^\mu = \gamma_u(1, \mathbf{u})$ and $v^\mu = \gamma_v(1, \mathbf{v})$. We call the frame of the light source S' and the

[1] Relatively fast, you may be tempted so say - a temptation you should resist.

[2] Ironically the precise example of particle number is NOT so objective when considering *quantum* particle states in full GR. But that topic is a potential discussion for muuuch later in the course - and is also part my PhD, if I finish it lol. In SR, the notion of the number of particles is well-defined and objective, worry not.

frame of the receiver S''.

In frame S, the 4-momentum of a photon emitted by the light source is given by $k^\mu = (\omega, \mathbf{k})$, where ω is the photon energy ($\hbar = 1$).

We use the fact that the inner product is, in general, invariant under Lorentz transformations between different inertial frames. This means a particular inner product in one frame must be equal to that same inner product in a different frame. In frame S we find

$$k^\mu u_\mu = -\gamma_u \omega (1 - u \cos\theta_u) \quad \text{and} \quad k^\mu v_\mu = -\gamma_v \omega (1 - v \cos\theta_v),$$

where θ_u and θ_v are the angles between the photon propagation direction and the direction of motion of respectively the light source and the receiver (and remember our metric sign convention is $(-1, 1, 1, 1)$).

However, in the rest frame S' of the light source we find

$$u'^\mu = (1, \mathbf{0}),\ k'^\mu = (\omega_s, \mathbf{k}') \quad \Longrightarrow \quad k'^\mu u'_\mu = -\omega_s,$$

and in the rest frame S'' of the receiver we obtain

$$v''^\mu = (1, \mathbf{0}),\ k''^\mu = (\omega_r, \mathbf{k}'') \quad \Longrightarrow \quad k''^\mu v''_\mu = -\omega_r.$$

We note that the photon energies in the different frames are different. Now, requiring $k^\mu u_\mu = k'^\mu u'_\mu$ as well as $k^\mu v_\mu = k''^\mu v''_\mu$ (from the invariance of the inner product) leads to two different expressions for ω. Equating those yields the Doppler relation between the photon frequencies in the source and receiver frames:

$$\omega = \frac{\omega_s}{\gamma_u (1 - u \cos\theta_u)} = \frac{\omega_r}{\gamma_v (1 - v \cos\theta_v)} \Longrightarrow \omega_r = \omega_s \frac{\gamma_v (1 - v \cos\theta_v)}{\gamma_u (1 - u \cos\theta_u)}.$$

The longitudinal Doppler effect you may be familiar with is recovered for something like $u = 0$ and $\theta_v = 0$, when a receiver and a source are moving away from each other with a relative velocity $v > 0$:

$$\omega_r = \omega_s \sqrt{\frac{1 - v}{1 + v}}.$$

The *transverse* Doppler effect is a bona fide relativistic effect. Consider a source emitting light while passing in front of a receiver: when a stationary receiver ($v = 0$) sees the light that the source emitted at their closest point, $\theta_u = \pi/2$, it is redshifted (the clock of the source ticks slow according to the receiver): $\omega_r = \omega_s / \gamma_u$.

Problem 5. The twin paradox

a) The short answer is that this scenario is not asymmetric because Bob undergoes acceleration at three different points: when blasting off from Earth, when having to slow down in order to turn around, and finally when coming back to Earth. Alice does not undergo any such accelerations.

   Bob thus switches between inertial reference frames and undergoes a sudden acceleration that changes his velocity from v to $-v$, making his reference frame briefly non-inertial. At the turnaround point, a discontinuous jump is seen in what Bob perceives as the time on Earth: although for him it normally looks like time on Earth is going slower during both the inbound and outbound journey, at the moment he turns around, from his point of view time on Earth suddenly fast-forwarded, resulting in his being younger when he meets Alice.

   The diagram shows this: the trip is split into two different inertial frames, while the person on Earth remains in one inertial frame.

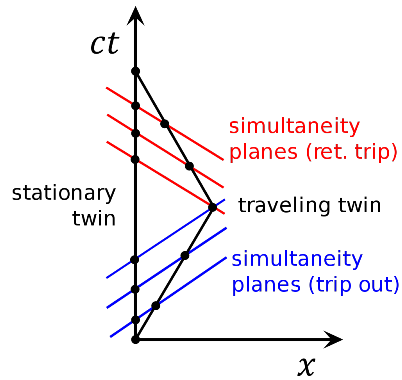


Figure 1: The twin paradox, with Bob's lines of simultaneity. (Taken again from good ol' Wikipedia). See also this nice illustration; and here is even more background.

b) Alice has grown older more rapidly than Bob by a factor $\gamma = (1 - v^2)^{-1/2}$, the usual time dilation factor. The factor is the same and constant during both Bob's outbound and inbound trips.

c) To use the coordinate transformation, we rewrite the invariant spacetime interval as

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = \eta_{\mu\nu} \frac{\partial x^\mu}{\partial x'^\rho} \frac{\partial x^\nu}{\partial x'^\sigma} dx'^\rho dx'^\sigma.$$

We know that $d\mathbf{x}' = 0$ for Bob's path in the new coordinates, so only the dt' terms matter:

$$d\tau^2 = -ds^2 = \left(\frac{\partial t}{\partial t'}\right)^2 dt'^2 - \left(\frac{\partial x}{\partial t'}\right)^2 dt'^2 = \left(1 - (v - gt')^2\right) dt'^2,$$

which leads to Bob's proper time (using $t' = t$)

$$d\tau = \int \sqrt{1 - (v - gt)^2}\, dt$$

The integrand is always smaller than 1 (except at the turnaround point, where $v = gt$), so Bob's proper time $\Delta\tau$ is less than Alice's Δt. Note how the integral shows that the time difference between the twins is built up continuously. This is consistent with the results in parts a) and b).

Indeed, we may use the result above to argue the paradox is resolved by noting the inevitable asymmetry between Alice and Bob: if Bob wants to return to Alice, his spacetime path must 'bend back', i.e. at some point his velocity needs to be reversed, which requires an acceleration. As soon as there is an acceleration in the game (and necessarily, Bob feels a force), Bob cannot claim to be in an inertial frame like Alice.

Notice how the integral for Bob's proper time just yields the expected time dilation $\Delta t/\gamma$ when his acceleration is set to zero – in that case the situation stays symmetrical and Bob sees Alice's time tick slower, but he never returns to her.

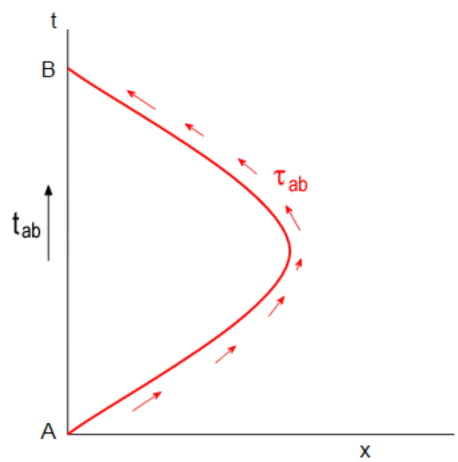


Figure 2: The twin paradox, with Bob decelerating at a constant rate.
Click here for the image source.