FYS4160 - General Relativity
Problem Set 2 Solutions
Spring 2024

These solutions are credited to Jake Gordin, who wrote them in the years 2020-23.

If you spot any typos, mistakes, don't hesitate to contact me at halvor.melkild@fys.uio.no. For any physics related question please use the forum at astro-discourse.uio.no.

The idea of these solutions is to give you a sense of what a 'model' answer should be, and they also elaborate on some discussions from the help sessions. I try to make them "pedagogical": i.e. hopefully comprehensive and most steps should be explained.

Problem 6. Tensors and stuff

a) Consider a general rank (k, l) tensor

$$T = T^{\mu_1 ... \mu_k}{}_{\nu_1 ... \nu_l} \; \hat{e}_{(\mu_1)} \otimes ... \otimes \hat{e}_{(\mu_k)} \otimes \hat{\theta}^{(\nu_1)} \otimes ... \otimes \hat{\theta}^{(\nu_l)}$$

- Retrieve any tensor component by letting the tensor act on the appropriate set of basis vectors and dual vectors, using the fact that these are defined such that $\hat{e}_{(\mu)}(\hat{\theta}^{(\nu)}) = \delta^\nu_\mu$ and $\hat{\theta}^{(\mu)}(\hat{e}_{(\nu)}) = \delta^\mu_\nu$:

$$\begin{aligned}
&T\left(\hat{\theta}^{(\mu_1)}, ..., \hat{\theta}^{(\mu_k)}, \hat{e}_{(\nu_1)}, ..., \hat{e}_{(\nu_l)}\right) \\
&= T^{\mu_1 ... \mu_k}{}_{\nu_1 ... \nu_l} \; \hat{e}_{(\mu_1)} \otimes ... \otimes \hat{e}_{(\mu_k)} \otimes \hat{\theta}^{(\nu_1)} \otimes ... \otimes \hat{\theta}^{(\nu_l)} \left(\hat{\theta}^{(\mu_1)}, ..., \hat{\theta}^{(\mu_k)}, \hat{e}_{(\nu_1)}, ..., \hat{e}_{(\nu_l)}\right) \\
&= T^{\mu_1 ... \mu_k}{}_{\nu_1 ... \nu_l} \left(\hat{e}_{(\mu_1)}(\hat{\theta}^{(\mu_1)}) \cdot ... \cdot \hat{e}_{(\mu_k)}(\hat{\theta}^{(\mu_k)}) \cdot \hat{\theta}^{(\nu_1)}(\hat{e}_{(\nu_1)}) \cdot ... \cdot \hat{\theta}^{(\nu_l)}(\hat{e}_{(\nu_l)})\right) \\
&= T^{\mu_1 ... \mu_k}{}_{\nu_1 ... \nu_l}
\end{aligned}$$

- Using this result and remembering that tensors are multilinear maps, we can immediately evaluate the action of T on k arbitrary dual vectors $\omega_{(i)} = \omega_{(i)\mu_i} \hat{\theta}^{(\mu_i)}$, $i = 1, ..., k$, and l arbitrary vectors $V^{(j)} = V^{(j)\nu_j} \hat{e}_{(\nu_j)}$, $j = 1, ..., l$:

$$\begin{aligned}
&T\left(\omega_{(1)}, ..., \omega_{(k)}, V^{(1)}, ..., V^{(l)}\right) \\
&= T\left(\omega_{(1)\mu_1}\hat{\theta}^{(\mu_1)}, ..., \omega_{(k)\mu_k}\hat{\theta}^{(\mu_k)}, V^{(1)\nu_1}\hat{e}_{(\nu_1)}, ..., V^{(l)\nu_l}\hat{e}_{(\nu_l)}\right) \\
&= T^{\mu_1 ... \mu_k}{}_{\nu_1 ... \nu_l} \omega_{(1)\mu_1} ... \omega_{(k)\mu_k} V^{(1)\nu_1} ... V^{(l)\nu_l} \hat{e}_{(\mu_1)} \otimes ... \otimes \hat{e}_{(\mu_k)} \\
&\quad \otimes \hat{\theta}^{(\nu_1)} \otimes ... \otimes \hat{\theta}^{(\nu_l)} \left(\hat{\theta}^{(\mu_1)}, ..., \hat{\theta}^{(\mu_k)}, \hat{e}_{(\nu_1)}, ..., \hat{e}_{(\nu_l)}\right) \\
&= T^{\mu_1 ... \mu_k}{}_{\nu_1 ... \nu_l} \omega_{(1)\mu_1} ... \omega_{(k)\mu_k} V^{(1)\nu_1} ... V^{(l)\nu_l}
\end{aligned}$$

b) Consider a rank (0,2) tensor S and a rank $(2, 0)$ tensor T.

- Write S and T as linear combinations of an arbitrary choice of basis vectors in the respective vector space:

$$S = S_{\mu\nu} \; \hat{\theta}^{(\mu)} \otimes \hat{\theta}^{(\nu)} \qquad T = T^{\mu\nu} \; \hat{e}_{(\mu)} \otimes \hat{e}_{(\nu)}.$$

Now anti-symmetrize the *components* of S, and introduce $\tilde{S}$ as the tensor that has these components (when retaining the same arbitrary basis vectors that you used to decompose S). In the same way, construct a tensor $\tilde{T}$ from T:

$$\tilde{S} = \tilde{S}_{\mu\nu} \; \hat{\theta}^{(\mu)} \otimes \hat{\theta}^{(\nu)} \text{ with } \tilde{S}_{\mu\nu} = S_{[\mu\nu]} = \frac{1}{2}(S_{\mu\nu} - S_{\nu\mu})$$

$$\tilde{T} = \tilde{T}^{\mu\nu} \; \hat{e}_{(\mu)} \otimes \hat{e}_{(\nu)} \text{ with } \tilde{T}^{\mu\nu} = T^{[\mu\nu]} = \frac{1}{2}(T^{\mu\nu} - T^{\nu\mu}).$$

- Show that $\tilde{T}(S) = T(\tilde{S}) = \tilde{T}(\tilde{S}) = \tilde{S}(\tilde{T})$:

$$\begin{aligned}\tilde{T}(S) &= T^{[\mu\nu]}\ \hat{e}_{(\mu)} \otimes \hat{e}_{(\nu)} \left(S_{\rho\sigma}\ \hat{\theta}^{(\rho)} \otimes \hat{\theta}^{(\sigma)} \right) \\ &= T^{[\mu\nu]} S_{\rho\sigma}\ \delta^{\rho}_{\mu}\delta^{\sigma}_{\nu} = T^{[\mu\nu]} S_{\mu\nu}\end{aligned}$$

  Similarly, we find

$$\begin{aligned}\tilde{T}(\tilde{S}) &= T^{[\mu\nu]} S_{[\mu\nu]} \\ T(\tilde{S}) &= T^{\mu\nu} S_{[\mu\nu]} \\ \tilde{S}(\tilde{T}) &= S_{[\mu\nu]} T^{[\mu\nu]}.\end{aligned}$$

  As we have already seen repeatedly before, $T^{[\mu\nu]} S_{\mu\nu} = T^{[\mu\nu]} S_{[\mu\nu]} = T^{\mu\nu} S_{[\mu\nu]}$. (You can show this by explicitly writing out the definition of the anti-symmetric brackets.)

Problem 7. Democracy but make it physics

For extra reading, refer to Carroll's discussion on pg. 48-50, and also to Schutz, pg. 171-173.

- *The weak equivalence principle (WEP) (Galilean equivalence principle).* Statement: The local effects of motion in a gravitational field are indistinguishable from those of an accelerated observer in flat spacetime. This is a fancy way of saying that one's inertial mass and gravitational mass are identical. The test for this is then obvious: drop two things with different weights off a tower (preferably a tower in Italy, maybe Pisa?) and see which hits the ground first. If they hit the ground at the same time, it means they experience the same acceleration, which means the inertial mass is equivalent to the gravitational one. Spoiler: they hit the ground at the same time.

- *The Einstein equivalence principle (EEP).* Statement: (1) the WEP holds and (2) in small enough regions of spacetime, the laws of physics reduce to those of special relativity - it is impossible to detect the existence of a gravitational field by means of local, non-gravitational experiments. This potentially sounds confusing, so here's another way of phrasing it: if the WEP says you can't tell if particles are freely-falling in a gravitational field or in an accelerated frame, the EEP says that you can't tell the difference between a freely-falling "frame" and an accelerated one.

  You can think of this as upgrading the inertial/gravitational equivalence to reference frames themselves. Two caveats: the first is that the experiment you do must itself be non-gravitational in nature; and second, you need to be in a small enough spacetime region. Gravitational redshift is a consequence of the WEP, since the change in wavelength due to motion should then be equivalent to that caused by free fall in a gravitational field.

  Bonus material: there is some controversy as to the precision of the Einstein's original formulation and whether tests actually test the EEP. There is similarly a debate still on what truly distinguishes a gravitational experiment from a non-gravitational one. For an interesting but pretty technical discussion of this, see `https://arxiv.org/pdf/gr-qc/0103067.pdf`.

- *The strong equivalence principle (SEP).* Statement: The outcome of any local experiment in a freely falling laboratory is independent of the velocity of the laboratory and its location in spacetime. This is the only form of the equivalence principle that applies to self-gravitating objects (such as stars), which have substantial internal gravitational interactions. Since the SEP states that all inertial frames with gravity are equivalent to accelerated frames without gravity, it requires that the gravitational constant be the same everywhere in the universe (or else there would be inertial gravitational frames that differ from non-gravitational accelerating ones). Possible tests include testing for time-variations of G, testing gravity theories that predict differing gravitational strengths in different regions (so-called fifth force theories), and so on. The aforementioned would test the SEP but not the other EPs.

Problem 8. Geodesic equation

The gist of this question is to show that our earlier expression for the Christoffel symbols, as acquired from the geodesic equation, is equivalent to the canonical form. Basically you want to show this:

$$\Gamma^{\mu}_{\rho\sigma} = \frac{\partial x^{\mu}}{\partial \xi^{\tau}} \frac{\partial^2 \xi^{\tau}}{\partial x^{\rho} \partial x^{\sigma}} = \frac{1}{2} g^{\mu\nu} \left(g_{\nu\rho,\sigma} - g_{\rho\sigma,\nu} + g_{\sigma\nu,\rho} \right). \tag{1}$$

There are (at least) three different ways to do this. For your edification and comfort, I'll sketch out all three: the one where you use the inverse metric, the one where you don't, and the "sneaky" way.

Method 1: grind your indices in the index grinder (no metric inverse)

We start with the RHS of eq. (1), and insert the relation

$$g_{\mu\nu} = \eta_{\alpha\beta} \frac{\partial \xi^{\alpha}}{\partial x^{\mu}} \frac{\partial \xi^{\beta}}{\partial x^{\nu}}.$$

Take care to keep track of indices which are dummy ones and those that are not: in the above, α, β are dummy indices (they can be swapped with any symbol, as long as they "cancel" overall). Note also the notation shorthand: $_{,\mu} \equiv \partial_{\mu} \equiv \partial / \partial x^{\mu}$. This is used often (I personally prefer ∂_{μ} but to each their own). We hence have:

$$\begin{aligned}
\frac{1}{2} g^{\mu\nu} \left(g_{\nu\rho,\sigma} - g_{\rho\sigma,\nu} + g_{\sigma\nu,\rho} \right) &= \frac{1}{2} g^{\mu\nu} \left(\eta_{\alpha\beta} \partial_{\sigma} (\partial_{\nu} \xi^{\alpha} \partial_{\rho} \xi^{\beta}) - \eta_{\alpha\beta} \partial_{\nu} (\partial_{\rho} \xi^{\alpha} \partial_{\sigma} \xi^{\beta}) + \eta_{\alpha\beta} \partial_{\rho} (\partial_{\sigma} \xi^{\alpha} \partial_{\nu} \xi^{\beta}) \right) \\
&= \frac{1}{2} g^{\mu\nu} \eta_{\alpha\beta} \left(\partial_{\sigma} \partial_{\nu} \xi^{\alpha} \partial_{\rho} \xi^{\beta} + \partial_{\nu} \xi^{\alpha} \partial_{\sigma} \partial_{\rho} \xi^{\beta} - \partial_{\nu} \partial_{\rho} \xi^{\alpha} \partial_{\sigma} \xi^{\beta} \right. \\
&\qquad \left. - \partial_{\rho} \xi^{\alpha} \partial_{\nu} \partial_{\sigma} \xi^{\beta} + \partial_{\rho} \partial_{\sigma} \xi^{\alpha} \partial_{\nu} \xi^{\beta} + \partial_{\sigma} \xi^{\alpha} \partial_{\rho} \partial_{\nu} \xi^{\beta} \right) \\
&= g^{\mu\nu} \eta_{\alpha\beta} \partial_{\nu} \xi^{\alpha} \partial_{\sigma} \partial_{\rho} \xi^{\beta} \\
&= g^{\mu\nu} \left(g_{\lambda\tau} \frac{\partial x^{\lambda}}{\partial \xi^{\alpha}} \frac{\partial x^{\tau}}{\partial \xi^{\beta}} \right) \frac{\partial \xi^{\alpha}}{\partial x^{\nu}} \partial_{\sigma} \partial_{\rho} \xi^{\beta} \\
&= g^{\mu\nu} \left(g_{\lambda\tau} \delta^{\lambda}_{\nu} \frac{\partial x^{\tau}}{\partial \xi^{\beta}} \right) \partial_{\sigma} \partial_{\rho} \xi^{\beta} \\
&= \delta^{\mu}_{\tau} \frac{\partial x^{\tau}}{\partial \xi^{\beta}} \partial_{\sigma} \partial_{\rho} \xi^{\beta} \\
&= \frac{\partial x^{\mu}}{\partial \xi^{\beta}} \frac{\partial^2 \xi^{\beta}}{\partial x^{\rho} \partial x^{\sigma}}.
\end{aligned}$$

This is the result we want.

A quick note about writing out $\eta_{\alpha\beta}$ in terms of $g_{\alpha\beta}$ in line 4: from the invariance of the spacetime interval in both coordinate systems, we know that

$$\begin{aligned}
ds^2 &= g_{\mu\nu} dx^{\mu} dx^{\nu} = \eta_{\alpha\beta} d\xi^{\alpha} d\xi^{\beta} \\
&= g_{\mu\nu} \frac{\partial x^{\mu}}{\partial \xi^{\alpha}} \frac{\partial x^{\nu}}{\partial \xi^{\beta}} d\xi^{\alpha} d\xi^{\beta},
\end{aligned}$$

where the second line makes use of the coordinate transformation $dx^\mu = (\partial x^\mu/\partial \xi^\nu)\, d\xi^\nu$. This allows us to identify

$$\eta_{\alpha\beta} = g_{\mu\nu}\frac{\partial x^\mu}{\partial \xi^\alpha}\frac{\partial x^\nu}{\partial \xi^\beta}.$$

Method 2: still on the sigma index grindset, but with the inverse metric

The same as before, only we use the inverse metric.

$$g^{\mu\nu} = \eta^{\alpha\beta}\frac{\partial x^\mu}{\partial \xi^\alpha}\frac{\partial x^\nu}{\partial \xi^\beta}. \tag{2}$$

If you're wondering how we can just write this, I refer you to the end of the solutions, where I derive this in detail (you can also just like ... move all the indices upstairs – but it's important to *know* you can just do that). Now, we insert this and write:

$$\begin{aligned}
\frac{1}{2}g^{\mu\nu}\left(g_{\nu\rho,\sigma} - g_{\rho\sigma,\nu} + g_{\sigma\nu,\rho}\right) &= \frac{1}{2}g^{\mu\nu}\left(\eta_{\alpha\beta}\partial_\sigma(\partial_\nu\xi^\alpha\partial_\rho\xi^\beta) - \eta_{\alpha\beta}\partial_\nu(\partial_\rho\xi^\alpha\partial_\sigma\xi^\beta) + \eta_{\alpha\beta}\partial_\rho(\partial_\sigma\xi^\alpha\partial_\nu\xi^\beta)\right)\\
&= \frac{1}{2}g^{\mu\nu}\eta_{\alpha\beta}\left(\partial_\sigma\partial_\nu\xi^\alpha\partial_\rho\xi^\beta + \partial_\nu\xi^\alpha\partial_\sigma\partial_\rho\xi^\beta - \partial_\nu\partial_\rho\xi^\alpha\partial_\sigma\xi^\beta\right.\\
&\qquad\qquad \left. -\partial_\rho\xi^\alpha\partial_\nu\partial_\sigma\xi^\beta + \partial_\rho\partial_\sigma\xi^\alpha\partial_\nu\xi^\beta + \partial_\sigma\xi^\alpha\partial_\rho\partial_\nu\xi^\beta\right)\\
&= \eta^{\rho\sigma}\frac{\partial x^\mu}{\partial \xi^\rho}\frac{\partial x^\nu}{\partial \xi^\sigma}\eta_{\alpha\beta}\partial_\nu\xi^\alpha\partial_\sigma\partial_\rho\xi^\beta\\
&= \eta^{\rho\sigma}\eta_{\alpha\beta}\frac{\partial x^\mu}{\partial \xi^\rho}\frac{\partial x^\nu}{\partial \xi^\sigma}\frac{\partial \xi^\alpha}{\partial x^\nu}\frac{\partial^2\xi^\beta}{\partial x^\rho\partial x^\sigma}\\
&= \eta^{\rho\sigma}\eta_{\alpha\beta}\frac{\partial x^\mu}{\partial \xi^\rho}\delta^\alpha_\sigma\frac{\partial^2\xi^\beta}{\partial x^\rho\partial x^\sigma}\\
&= \frac{\partial x^\mu}{\partial \xi^\beta}\frac{\partial^2\xi^\beta}{\partial x^\rho\partial x^\sigma}\,.
\end{aligned}$$

One less line than method 1, we love to see it.

Method 3: sneak 100

Alternatively, we could have started by noting that

$$\frac{\partial^2\xi^\tau}{\partial x^\rho\partial x^\sigma} = \frac{\partial \xi^\tau}{\partial x^\mu}\Gamma^\mu_{\ \rho\sigma},$$

which implies

$$\begin{aligned}
g_{\mu\nu,\lambda} &= \partial_\lambda\left(\eta_{\alpha\beta}\partial_\mu\xi^\alpha\partial_\nu\xi^\beta\right)\\
&= \partial_\nu\xi^\beta\eta_{\alpha\beta}\left(\partial_\lambda\partial_\mu\xi^\alpha\right) + \partial_\mu\xi^\alpha\eta_{\alpha\beta}\left(\partial_\lambda\partial_\nu\xi^\beta\right)\\
&= \underbrace{\partial_\nu\xi^\beta\eta_{\alpha\beta}\frac{\partial\xi^\alpha}{\partial x^\rho}}_{g_{\rho\nu}}\Gamma^\rho_{\ \lambda\mu} + \underbrace{\partial_\mu\xi^\alpha\eta_{\alpha\beta}\frac{\partial\xi^\beta}{\partial x^\rho}}_{g_{\nu\rho}}\Gamma^\rho_{\ \lambda\nu}\\
&= \Gamma^\rho_{\ \lambda\mu}g_{\rho\nu} + \Gamma^\rho_{\ \lambda\nu}g_{\mu\rho}\,.
\end{aligned}$$

Inserting this relation into the RHS of eq. (1), we find:

$$
\begin{aligned}
\frac{1}{2}g^{\mu\nu}\left(g_{\nu\rho,\sigma} - g_{\rho\sigma,\nu} + g_{\sigma\nu,\rho}\right) &= \frac{1}{2}g^{\mu\nu}\left(\Gamma^{\tau}_{\sigma\nu}g_{\rho\tau} + \Gamma^{\tau}_{\sigma\rho}g_{\nu\tau} - \Gamma^{\tau}_{\sigma\nu}g_{\rho\tau} - \Gamma^{\tau}_{\rho\nu}g_{\sigma\tau} + \Gamma^{\tau}_{\sigma\rho}g_{\nu\tau} + \Gamma^{\tau}_{\rho\nu}g_{\sigma\tau}\right) \\
&= g^{\mu\nu}\Gamma^{\tau}_{\sigma\rho}g_{\nu\tau} \\
&= \Gamma^{\mu}_{\sigma\rho}\,.
\end{aligned}
$$

which equals our starting point, as by definition. Note that from line 1 to line 2 I cancelled like terms, leaving only two terms remaining that are equal (which gets rid of the factor of $\frac{1}{2}$).

Appendix: the inverse of $g_{\mu\nu}$ in detail

Earlier I said we may write $g^{\mu\nu}$ as

$$g^{\mu\nu} = \eta^{\alpha\beta}\frac{\partial x^{\mu}}{\partial \xi^{\alpha}}\frac{\partial x^{\nu}}{\partial \xi^{\beta}}. \tag{3}$$

"But Jake, how do we *know* you can just move the indices upstairs and then swap the fractions?", I hear you ask. Well, grab a cup of tea and settle down - here I'll go through in *excruciating* detail how one derives this.

The basic logic is actually that of finding inverses in regular algebra, only with lots of indices. First things first: from the definition of the metric tensor, we can write

$$g^{\lambda\nu}g_{\nu\mu} = \delta^{\lambda}{}_{\mu} \tag{4}$$

(I have staggered the indices for some people's peace of mind; appreciate the effort it takes to write in the additional curly brackets).

Next, we substitute in eq. (3) into eq. (4)

$$\begin{aligned} g^{\lambda\nu}\eta_{\alpha\beta}\frac{\partial \xi^{\alpha}}{\partial x^{\mu}}\frac{\partial \xi^{\beta}}{\partial x^{\nu}} &= \delta^{\lambda}{}_{\mu} \\ g^{\lambda\nu}\eta_{\alpha\beta}\frac{\partial \xi^{\alpha}}{\partial x^{\mu}}\frac{\partial \xi^{\beta}}{\partial x^{\nu}} - \delta^{\lambda}{}_{\mu} &= 0 \\ \eta_{\alpha\beta}\left(g^{\lambda\nu}\frac{\partial \xi^{\alpha}}{\partial x^{\mu}}\frac{\partial \xi^{\beta}}{\partial x^{\nu}} - \delta^{\alpha}{}_{\mu}\delta^{\beta}{}_{\nu}\eta^{\lambda\nu}\right) &= 0 \end{aligned}$$

Okay, a lot happened in those three lines - don't panic! Let's go through it step by step.

From line 2 to 3, I took out an $\eta_{\alpha\beta}$ as a common factor. So the first term in line 3 is the same as it was above, without the $\eta_{\alpha\beta}$.

The second term in line 3 is what happens when you take out $\eta_{\alpha\beta}$ as a common factor from a Kronecker delta. We know that a tensor multiplied with its inverse gives you the Kronecker, so if we take out a common factor of any tensor from the Kronecker, you must have the inverse left, in this case $\eta^{\lambda\mu}$. Then the extra two Kroneckers, $\delta^{\alpha}{}_{\mu}\delta^{\beta}{}_{\nu}$, are there to preserve the index convention (ultimately, each term must have one λ upstairs and one μ downstairs!). Check for yourself that this is true, try go back from line 3 to line 2.

Now, since $\eta_{\alpha\beta}$ is a common factor, we know that the stuff inside the brackets must equal zero. Therefore,

$$g^{\lambda\nu}\frac{\partial \xi^{\alpha}}{\partial x^{\mu}}\frac{\partial \xi^{\beta}}{\partial x^{\nu}} - \delta^{\alpha}{}_{\mu}\delta^{\beta}{}_{\nu}\eta^{\lambda\nu} = 0.$$

Let's do something which will seem strange now but will make our lives a bit easier: multiply everything by $\delta^{\mu}{}_{\lambda}$:

$$\delta^{\mu}{}_{\lambda} g^{\lambda\nu} \frac{\partial \xi^{\alpha}}{\partial x^{\mu}} \frac{\partial \xi^{\beta}}{\partial x^{\nu}} - \delta^{\mu}{}_{\lambda} \delta^{\alpha}{}_{\mu} \delta^{\beta}{}_{\nu} \eta^{\lambda\nu} = 0.$$

Contract ALL the Kronecker deltas (exercise for the reader :p) and you get:

$$g^{\mu\nu} \frac{\partial \xi^{\alpha}}{\partial x^{\mu}} \frac{\partial \xi^{\beta}}{\partial x^{\nu}} = \eta^{\alpha\beta}.$$

We're almost there! We have a transformation for the η coordinate system, as the inverse. But we wanted the inverse of g. So, let's contract each term with the factor

$$\frac{\partial x^{\tau}}{\partial \xi^{\beta}} \frac{\partial x^{\sigma}}{\partial \xi^{\alpha}}.$$

and therefore we have

$$\begin{aligned} g^{\mu\nu} \frac{\partial \xi^{\alpha}}{\partial x^{\mu}} \frac{\partial \xi^{\beta}}{\partial x^{\nu}} \frac{\partial x^{\tau}}{\partial \xi^{\beta}} \frac{\partial x^{\sigma}}{\partial \xi^{\alpha}} &= \eta^{\alpha\beta} \frac{\partial x^{\tau}}{\partial \xi^{\beta}} \frac{\partial x^{\sigma}}{\partial \xi^{\alpha}} \\ g^{\mu\nu} \underbrace{\frac{\partial \xi^{\alpha}}{\partial x^{\mu}} \frac{\partial x^{\sigma}}{\partial \xi^{\alpha}}}_{=\delta^{\sigma}{}_{\mu}} \underbrace{\frac{\partial \xi^{\beta}}{\partial x^{\nu}} \frac{\partial x^{\tau}}{\partial \xi^{\beta}}}_{=\delta^{\tau}{}_{\nu}} &= \eta^{\alpha\beta} \frac{\partial x^{\tau}}{\partial \xi^{\beta}} \frac{\partial x^{\sigma}}{\partial \xi^{\alpha}} \\ g^{\tau\sigma} &= \eta^{\alpha\beta} \frac{\partial x^{\tau}}{\partial \xi^{\beta}} \frac{\partial x^{\sigma}}{\partial \xi^{\alpha}} \end{aligned}$$

Swapping dummy indices τ, σ for letters which are a little less obnoxious, we find

$$g^{\mu\nu} = \eta^{\alpha\beta} \frac{\partial x^{\mu}}{\partial \xi^{\beta}} \frac{\partial x^{\nu}}{\partial \xi^{\alpha}}$$

as required.