FYS4160 - General Relativity
Problem Set 3 Solutions
Spring 2024

These solutions are credited to Jake Gordin, who wrote them in the years 2020-23.

If you spot any typos, mistakes, don't hesitate to contact me at halvor.melkild@fys.uio.no. For any physics related question please use the forum at astro-discourse.uio.no.

The idea of these solutions is to give you a sense of what a 'model' answer should be, and they also elaborate on some discussions from the help sessions. I try to make them "pedagogical": i.e. hopefully comprehensive and most steps should be explained.

Problem 9. Geodesic equation

(a) First things first - why doesn't the variation work for photons? Varying the action

$$S = \int d\tau = \int \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}d\tau$$

will lead to the geodesic equation. If we do the first step in the variation, we find that

$$\begin{aligned}\delta S &= \int \delta\sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}d\tau \\ &= \int \frac{1}{2}\left(-g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right)^{-1/2} \delta\left(-g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right)d\tau.\end{aligned}$$

This doesn't work for photons since the spacetime interval for null trajectories is $ds^2 = 0$, which means we're dividing by zero with the $[(\text{stuff})^{-1/2}]$ term.

Also: what is the most general choice of λ, the most general affine parameter, that will still keep the geodesic equation in its normal form? What this means is: what can λ be and still have the action's variation result in

$$\ddot{X}^\mu + \Gamma^\mu{}_{\rho\sigma}\dot{X}^\rho\dot{X}^\sigma = 0,$$

where a dot denotes differentiation with respect to this affine parameter, λ.

We can start by parameterising the affine parameter as a function, since we want the most general choice for it. We do this as a of function the proper time, $\lambda = \lambda(\tau)$, the variable of integration in the original action. Then we'll see what values of λ keep the geodesic equation invariant.

The generic coordinate variable X^μ is a function of λ; and so with our parameterisation of the geodesic curve $X^\mu(\tau) = X^\mu(\lambda(\tau))$, we can calculate the first and second derivatives by differentiation with respect to τ:

$$\frac{d}{d\tau} = \frac{d}{d\lambda}\frac{d\lambda}{d\tau} \quad \text{and} \quad \frac{d^2}{d\tau^2} = \frac{d^2\lambda}{d\tau^2}\frac{d}{d\lambda} + \left(\frac{d\lambda}{d\tau}\right)^2\frac{d^2}{d\lambda^2}$$

I used the chain and product rule, and to get the second term in the second derivative we use the result from the first derivative.

Since our geodesic varies with respect to λ, we need to invert the above and express it in terms of the λ derivatives:

$$\frac{d}{d\lambda} = \frac{d}{d\tau}\frac{d\tau}{d\lambda} \quad \text{and} \quad \frac{d^2}{d\lambda^2} = \left(\frac{d\tau}{d\lambda}\right)^2 \frac{d^2}{d\tau^2} - \left(\frac{d\tau}{d\lambda}\right)^2 \frac{d}{d\lambda}\frac{d^2\lambda}{d\tau^2}$$

The geodesic equation becomes, considering path terms only

$$\ddot{X}^\mu \implies \left(\frac{d\tau}{d\lambda}\right)^2 \frac{d^2x^\mu}{d\tau^2} - \frac{d^2\lambda}{d\tau^2}\left(\frac{d\tau}{d\lambda}\right)^2 \frac{dx^\mu}{d\lambda}$$
$$\dot{X}^\rho \dot{X}^\sigma \implies \left(\frac{d\tau}{d\lambda}\right)^2 \frac{d\dot{x}^\rho}{d\tau}\frac{d\dot{x}^\sigma}{d\tau}$$

After dividing by $(d\tau/d\lambda)^2$ (non-zero for an monotonic parameter transformation), we get left with

$$\frac{d^2x^\mu}{d\tau^2} - \frac{d^2\lambda}{d\tau^2}\frac{dx^\mu}{d\lambda} + \Gamma^\mu{}_{\rho\sigma}\frac{d\dot{x}^\rho}{d\tau}\frac{d\dot{x}^\sigma}{d\tau} = 0.$$

As such, we require

$$\frac{d^2\lambda}{d\tau^2} = 0 \quad \iff \quad \lambda = a\tau + b,$$

with a, b constants, to retain the geodesic's canonical form. This is the most general form in which λ can be and still give us the correct geodesic equation.

(b) The action now reads

$$\delta S = \delta\left(-g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right) d\tau = 0$$

which follows the derivation in the book, p. 106-108, only now without the "division by zero" problems (hurray!). This geodesic equation is valid for photons ($ds^2 = 0$) following a curve that satisfies this equation, $\mathcal{K} = g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0$. I won't include the details here because it's step-for-step in Carroll.

Problem 10. Prolate spherical coordinates

Problem 2.7 in Carroll reads:

Prolate spheroidal coordinates can be used to simplify the Kepler problem in celestial mechanics. They are related to the usual cartesian coordinates (x, y, z) of Euclidean three-space by

$$x = \sinh\chi \sin\theta \cos\phi$$
$$y = \sinh x \sin\theta \sin\phi$$
$$z = \cosh x \cos\theta$$

Restrict your attention to the plane $y = 0$ and answer the following questions.

(a) What is the coordinate transformation matrix $\partial x^\mu / \partial x^{v'}$ relating (x, z) to (χ, θ)?
(b) What does the line element ds^2 look like in prolate spheroidal coordinates?

(a) In this coordinate system, the y-plane corresponds to $\phi = 0$. Hence,

$$\begin{aligned} x &= \sinh\chi \sin\theta \\ z &= \cosh\chi \cos\theta . \end{aligned}$$

The coordinate transformation matrix becomes

$$\frac{\partial x^{\mu}}{\partial x^{\nu'}} = \begin{pmatrix} \frac{\partial x}{\partial \chi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial z}{\partial \chi} & \frac{\partial z}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cosh\chi \sin\theta & \sinh\chi\cos\theta \\ \sinh\chi\cos\theta & -\cosh\chi\sin\theta \end{pmatrix}.$$

(b) For the line element, we find the following:

$$\begin{aligned} dx &= \frac{\partial x}{\partial \chi}\, d\chi + \frac{\partial x}{\partial \theta}\, d\theta = \cosh\chi\sin\theta\, d\chi + \sinh\chi\cos\theta\, d\theta \\ dz &= \frac{\partial z}{\partial \chi}\, d\chi + \frac{\partial z}{\partial \theta}\, d\theta = \sinh\chi\cos\theta\, d\chi - \cosh\chi\sin\theta\, d\theta \end{aligned}.$$

Using $\cosh^2 x - \sinh^2 x = 1$, the end result is

$$ds^2 = dx^2 + dz^2 = \left(\sinh^2\chi + \sin^2\theta\right)\left(d\chi^2 + d\theta^2\right).$$

Problem 11. GPS satellite

(a) Special relativity implies that the clocks on a moving satellite tick slower than on Earth. In particular, $\Delta\tau_{sat} = \Delta\tau_{Earth}/\gamma$, and using

$$1/\gamma = (1 - v^2)^{(1/2)} \simeq 1 - (v/c)^2/2 = 1 - 4.3 \times 10^{-11},$$

we find for $\Delta\tau_{Earth} = 1$ day the time difference

$$(\Delta\tau_{sat} - \Delta\tau_{Earth})_{SR} = \left(\frac{1}{\gamma} - 1\right) \cdot 86\,400 \text{ s} = -3.7\ \mu\text{s}.$$

As expected, less time passed on the satellite than on Earth.

You may be justifiably worried about the fact that the satellite is in orbit and must undergo acceleration as its velocity constantly changes direction. Note however that the magnitude of its velocity does not change. So at each point of the satellite's orbit, it is instantaneously in an inertial frame, where clocks run slower than on Earth by a fixed factor γ. This is spelled out in the Q&A below, borrowed from Perimeter Institute.

FREQUENTLY ASKED QUESTIONS

Q - At different positions in its orbit, a GPS satellite will have differing speeds relative to different GPS receivers. Given this, do we need to adjust the speed used in the equation for time dilation to account for this variation?

A - In principle, we do need to use a different value for *v* in Equation 1 depending on the precise speed of a given satellite relative to a particular receiver. However, the speed of the satellites (3874 m/s) is much larger than the speed of a GPS receiver as it moves with Earth's rotation (465 m/s at the equator). Differences in the values of the relative speed between a satellite and a receiver result in variations in the amount of time dilation of just 1% at most and so are insignificant for the current accuracy of the GPS.

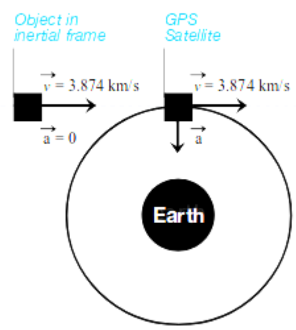
Q - GPS satellites are in orbit and so are accelerating. They are not in inertial reference frames. Similarly, GPS receivers are accelerating due to Earth's rotation and so are also not in inertial frames. Given this, how can we use special relativity, which primarily deals with inertial frames, to calculate the amount of time dilation?

A - The reason we can use this theory is that the acceleration of GPS receivers (0.034 m/s^2) is so small that we can ignore it. Over the course of one second, the acceleration changes each receiver's speed by just 0.034 m/s. For a receiver at the equator, this is just 0.007% of its speed due to Earth's rotation. So, the effect the acceleration has on the amount of time dilation is at most only about 0.007% of the total value per day of 7 μs. This corresponds to just 0.0005 μs, a negligible effect.

Approximating GPS receivers as being in inertial frames, a GPS satellite moves at a speed of 3874 m/s relative to this frame. At each moment in time, it has an instantaneous velocity of 3874 m/s along its orbit.

Imagine a second object with the same velocity but which is not accelerating (see Figure 5). This object is in an inertial frame and so, using Equation 1, we can calculate that we see its clock running slow by 8.3×10^{-11} s per second. The GPS satellite shares the same instantaneous motion and so we will also see its clock running slow by the same amount. In the next instant, the satellite clock shares the same motion as a third object moving at 3874 km/s in a slightly different inertial frame. So, its clock runs slow by the same amount as in the previous instant.

Continuing this process over the satellite's entire orbit, we find that the satellite's clock runs slow by 8.3×10^{-11} s per second throughout its orbit. We can use special relativity at each instant of the satellite's motion and then add up all of the amounts of time dilation to calculate the total amount. Even though the satellite is accelerating, by comparing it to other objects in inertial frames moving at the same instantaneous speeds, we can use special relativity to determine how slowly its clock runs.



(b) From general relativity we need to account for the effect of Earth's gravitational field on both clocks. In particular, we know that

$$\Delta\tau_{sat} = \Delta t\sqrt{1 + 2\phi(r_{sat})}$$
$$\Delta\tau_{Earth} = \Delta t\sqrt{1 + 2\phi(r_{Earth})},$$

where we used the gravitational potential (per unit of mass)

$$\phi(r) = -\frac{GM_{Earth}}{r}.$$

Δt denotes the amount of time that passed for an observer infinitely far away and unaffected by Earth's gravitational field. Furthermore, we know that $r_{Earth} = 6378$ km, $r_{sat} = r_{Earth} + 20\,000$ km and $M_{Earth} = 5.972 \times 10^{24}$ kg.

From each of the first two equations, we can extract Δt; equating the two expressions yields

$$\Delta\tau_{sat} = \Delta\tau_{Earth}\sqrt{\frac{1 + 2\phi(r_{sat})}{1 + 2\phi(r_{Earth})}}.$$

Note that ϕ is dimensionless in natural units. To make the units in the equation work out when using SI units, we need to make the replacement $\phi \to \phi/c^2$.

Let us approximate this to first order in ϕ/c^2:

$$\begin{aligned}\Delta\tau_{sat} &\simeq \Delta\tau_{Earth}(1+\phi(r_{sat})/c^2)(1-\phi(r_{Earth})/c^2)\\ &= \Delta\tau_{Earth}(1+\Delta\phi/c^2)+\mathcal{O}(\phi^2/c^4)),\end{aligned}$$

where

$$\Delta\phi/c^2 = (\phi(r_{sat})-\phi(r_{Earth}))/c^2 = 5.26\times 10^{-10}.$$

Then we find, with $\Delta\tau_{Earth} = 1$ day, the time difference

$$(\Delta\tau_{sat}-\Delta\tau_{Earth})_{GR} = \Delta\phi/c^2\cdot 86\,400\text{s} = 45.5\ \mu\text{s}.$$

As expected, the time on the surface of the Earth, i.e. deep inside the gravitational potential well of the Earth, goes slower than for the satellite.

The total time difference after one day on Earth is

$$\Delta\tau_{sat}-\Delta\tau_{Earth} = (-3.7+45.5)\ \mu\text{s} = 41.8\ \mu\text{s}.$$

Note that once we pick the satellite to be the frame we refer to as moving, we need to be consistent in this. So the value of $\Delta\tau_{sat}-\Delta\tau_{Earth}$ must be kept in that order. If both SR and GR have the same sign, you've mixed them up!

Finally, the value of 41.8 μs corresponds to a length scale of $c\cdot 41.8\ \mu\text{s}\simeq 12.5$ km, the scale of GPS errors after just one day without relativistic corrections. That is to say, your Foodora driver would be a good 12kms off your house without factoring in relativistic effects. (Thankfully the driver's phone does it for them).