FYS4160 - General Relativity
Problem Set 4 Solutions
Spring 2024

These solutions are credited to Jake Gordin, who wrote them in the years 2020-23.

If you spot any typos, mistakes, don't hesitate to contact me at halvor.melkild@fys.uio.no. For any physics related question please use the forum at astro-discourse.uio.no.

The idea of these solutions is to give you a sense of what a 'model' answer should be, and they also elaborate on some discussions from the help sessions. I try to make them "pedagogical": i.e. hopefully comprehensive and most steps should be explained.

Problem 12. Atlas for a 2-torus

T^2 is the direct product $S^1 \times S^1$ of two manifolds. So an atlas for those two circles, glued together, will then be the atlas for for the torus. The circle S^1 needs two charts, for example (this is by no means a unique choice) covering the circle with two overlapping domains $(0, 2\pi)$ and $(-\pi, \pi)$.

We can extend this to the torus, of which every point can be identified by two angular coordinates, one on the interior circle and one on the exterior one:

$$T^2 = \{(\theta, \phi) \mid 0 \leq \theta < 2\pi, 0 \leq \phi < 2\pi\}\,.$$

The torus is covered by the following four domains:

$$\begin{aligned}
U_1 &= \{(\theta, \phi) \mid 0 < \theta < 2\pi, 0 < \phi < 2\pi\} \\
U_2 &= \{(\theta, \phi) \mid -\pi < \theta < \pi, 0 < \phi < 2\pi\} \\
U_3 &= \{(\theta, \phi) \mid 0 < \theta < 2\pi, -\pi < \phi < \pi\} \\
U_4 &= \{(\theta, \phi) \mid -\pi < \theta < \pi, -\pi < \phi < \pi\}\,,
\end{aligned}$$

and the atlas is completed by a set of corresponding charts ($i = 1, 2, 3, 4$)

$$\phi_i : U_i \to \phi_i(U_i) \subset \mathbb{R}^2 : (\theta, \phi) \mapsto (x = \theta, y = \phi).$$

Just so we don't miss the forest for the trees here, because I know the notation and language is a bit formidable: all we're saying that we need two charts to cover a circle, and doing this with two cirlces will cover the torus. Each circle will have a different angle (see Carroll pg. 60-61 for extra details); we simply do this twice, for two different angles, which is what those four charts are.

Problem 13. Coordinate bases.

- For a coordinate system with some coordinate basis, the vector $V = V^\mu \partial_\mu$ has components V^μ. This is true no matter the basis.

- The transformation law between bases is given by

$$\partial_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu$$

  A position vector in spherical space, $\vec{r}$, can be written in terms of Euclidean basis vectors via

$$\vec{r} = r\cos\theta\sin\phi\hat{\imath} + r\sin\theta\sin\phi\hat{\jmath} + r\cos\phi\hat{k}.$$

Let us compute the first basis vector, $\mu' = r$,

$$\partial_{r'} = \frac{\partial x^\mu}{\partial r'}\partial_\mu = \frac{\partial x}{\partial r'}\partial_x + \frac{\partial y}{\partial r'}\partial_y + \frac{\partial z}{\partial r'}\partial_z$$
$$= \cos\theta\sin\phi\hat{\imath} + \sin\theta\sin\phi\hat{\jmath} + \cos\phi\hat{k}$$

The remaining basis vectors then turn out to be

$$\partial_{\theta'} = \frac{\partial\vec{r}}{\partial\theta} = -r\sin\theta\sin\phi\hat{\imath} + r\cos\theta\sin\phi\hat{\jmath}$$
$$\partial_{\phi'} = \frac{\partial\vec{r}}{\partial\phi} = r\cos\theta\cos\phi\hat{\imath} + r\sin\theta\cos\phi\hat{\jmath} - r\sin\phi\hat{k}$$

The reverse transformations, found in much the same way, are given by

$$\hat{\imath} = \cos\theta\sin\phi\partial_{r'} - \sin\theta\partial_{\theta'} + \cos\theta\cos\phi\partial_{\phi'}$$
$$\hat{\jmath} = \sin\theta\sin\phi\partial_{r'} + \cos\theta\partial_{\theta'} + \sin\theta\cos\phi\partial_{\phi'}$$
$$\hat{k} = \cos\phi\partial_{r'} - \sin\phi\partial_{\phi'}$$

- Carroll pg 63-65.

Problem 14. The Lie bracket.

- Vector fields on manifolds are examples of what's known as *derivations* on $C^\infty(M)$. By definition, derivations obey Leibniz's rule. Let $X, Y \in \mathfrak{X}(M)$, and let $f, g \in C^\infty(M)$, then

$$\begin{aligned}
[X,Y](fg) &= X(Y(fg)) - Y(X(fg)) \\
&= X\{gY(f) + fY(g)\} - Y\{gX(f) + fX(g)\} \\
&= g\{XY(f) - YX(f)\} + f\{XY(g) - YX(g)\} \\
&= g[X,Y](f) + f[X,Y](g).
\end{aligned}$$

  Hence $[X,Y]$ is a derivation $C^\infty(M) \to C^\infty(M)$, which makes it a vector field by definition. Note that in line 2, $Y(fg)) = gY(f) + fY(g)$ (and the same for X) since we already know X, Y are individually vector fields and therefore can be known to obey the product rule.

  Note that this a very abstract means of proof. There are other ways to show this. For example, by using the tangent bundle definition of vector fields, and demonstrating that $[X,Y]$ possesses those properties - linearity and obeys the product rule - would show it's a vector field. Another way is to express the Lie bracket in coordinate form and show it transforms as a vector field does (eq. 2.19 in Carroll). The important thing is less which method you use, and more that you have fun doing it (insofar as one can have fun proving Lie bracket identities - things like *going outside* and *meeting people* are overrated anyway).

- In the local coordinate basis associated with he tangent bundle, vector fields can be written $X = X^\mu\partial_\mu$ and $Y = Y^\mu\partial_\mu$. Then the Lie bracket can be computed as:

$$[X,Y] := (X(Y^\mu) - Y(X^\mu))\,\partial_\mu = (X^\nu\partial_\nu Y^\mu - Y^\nu\partial_\nu X^\mu)\,\partial_\mu$$

Since the basis chosen is ∂_μ, the *components* of the Lie bracket are thus

$$[X,Y]^\mu := X^\nu \partial_\nu Y^\mu - Y^\nu \partial_\nu X^\mu$$

- Follows immediately from the fact that partial derivatives commute.