FYS4160 - General Relativity
Problem Set 5 Solutions
Spring 2024

These solutions are credited to Jake Gordin, who wrote them in the years 2020-23.

If you spot any typos, mistakes, don't hesitate to contact me at halvor.melkild@fys.uio.no. For any physics related question please use the forum at astro-discourse.uio.no.

The idea of these solutions is to give you a sense of what a 'model' answer should be, and they also elaborate on some discussions from the help sessions. I try to make them "pedagogical": i.e. hopefully comprehensive and most steps should be explained.

Problem 15. The exterior derivative.

Before we get into the questions, I know Carroll has a rather basic introduction and doesn't really explain anything beyond the definitions he gives. In addition, he eschews the local coordinate definition of differential forms. To (somewhat) remedy the situation, I'll include here some extra information and resources, so that you don't have to accept entirely at face value what I write down later.

A differential form primer.

Carroll mentions differential forms, and then defines the wedge product and exterior derivative without much explanation of the forms themselves.

Before we get into p-forms and all that jazz, let's think about 1-forms (the easy bois). The set of 1-forms denoted by $dx^1, \ldots, dx^n$ is *defined* such that they form a basis for the 1-forms on $\mathbb{R}^n$, so any 1 -form ϕ may be expressed in the form

$$\phi = \sum_{i=1}^{n} f_i(x) dx^i$$

where $f_i(x)$ is some function. If a vector field v on $\mathbb{R}^n$ has the form

$$v(x) = \left(v^1(x), \ldots, v^n(x)\right)$$

then at any point $x \in \mathbb{R}^n$,

$$\phi_x(v) = \sum_{i=1}^{n} f_i(x) v^i(x).$$

Now, let's upgrade to p-forms. Firstly, the 1-forms on $\mathbb{R}^n$ are part of an *algebra*[1], called the algebra of differential forms on $\mathbb{R}^n$. The multiplication type in this algebra is called the wedge product, and it is anti-symmetric:

$$dx^i \wedge dx^j = -dx^j \wedge dx^i.$$

One consequence of this is that $dx^i \wedge dx^i = 0$ (alternatively, one can start with the construction of a wedge product in a way that takes $dx^i \wedge dx^i = 0$ as axiomatic, and uses this to prove the anti-symmetry relation). If each differential form ϕ contains $p dx^i$'s, the form is called a p-form. Functions alone are considered to be 0 -forms. A basis for the p-forms on $\mathbb{R}^n$ is given by the set

$$\left\{dx^{i_1} \wedge \cdots \wedge dx^{i_p} : 1 \leq i_1 < i_2 < \cdots < i_p \leq n\right\}.$$

[1] The precise properties of algebras in this context are a bit hectic to go into here, but the Wikipedia article introduces the basics `https://en.wikipedia.org/wiki/Algebra_over_a_field`.

Any p-form ϕ may then be expressed in the form

$$\phi = \sum_{|I|=p} f_I dx^{i_1} \wedge \cdots \wedge dx^{i_p}$$

where I ranges over all multi-indices $I = (i_1, \ldots, i_p)$ of length p. In a more condensed and useful fashion, one can write a p-form as

$$\phi = f_I dx^I$$

where the sum is omitted for brevity.

The notation just introduced here may at seem at odds with that in Carroll, when he defined the exterior derivative and wedge product. In the notation above, the exterior derivative is

$$d\phi = \sum_{|I|=p} df_I \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p};$$

the wedge product is merely

$$\phi \wedge \psi = \sum_{|I|=p} \sum_{|J|=q} f_I dx^I \wedge g_J dx^J.$$

For this problem, you can use either form of notation: with all the indices and weird anti-symmetric brackets and combinatoric stuff (as used in Carroll) or the local coordinate way, with condensed and sensible notation. No prizes for guessing which I think is best. Besides, this way is far more “user/GR student friendly”. My laziness aside, this way of writing differential forms also illuminate the conceptual similarity of the exterior derivative to other notions of taking derivatives.

That said, if you want a step-by-step guide on how to pass between the different notation, this post is rather useful:
https://math.stackexchange.com/questions/3185894/how-to-prove-leibniz-rule-for-exterior-derivative-usin 3187424#3187424. Feel free to ask questions about it on astro-discourse or email me if something is unclear.

Additionally, for information about differential forms and whatnot in general – if you have free time you simply must use on understanding differential geometry – these lecture notes are rather good, https://spot.colorado.edu/~jnc/lecture1.pdf. If you *really* have a lack of obtuse maths in your life, and want to remedy that, I also refer you to this textbook, http://www.math.u-szeged.hu/~stacho/mm/flanders.pdf (start from chapter III, page 35). For a more mathsey textbook, it’s actually decently written.

(a) With all the preamble stuff out of the way, let’s calculate $d(\omega \wedge \eta)$:

$$\begin{aligned} d(\omega \wedge \eta) &= d((f dx^I) \wedge (g dx^J)) \\ &= d(fg dx^I \wedge dx^J) \\ &= (g df + f dg) \wedge dx^I \wedge dx^J \\ &= (df \wedge dx^I) \wedge (g dx^J) + (-1)^p (f dx^I) \wedge (dg \wedge dx^J) \\ &= d\omega \wedge \eta + (-1)^p \omega \wedge d\eta \end{aligned}$$

In line 1 I used the fact that a p-form can be expressed as $\omega = f dx^I$, as discussed above; in line 2, note that f, g are merely functions and can be moved in front of the wedge products. Line 3 is an application of the chain rule and the fact that $d\omega = d(f dx^I) = df \wedge dx^I$, and line 4 uses the antisymmetry property of the ordering of forms, i.e. $dg \wedge dx^I = (-1)^p dx^I \wedge dg$.

(b) Let's take the "double exterior derivative" of a 0-form f first, since that's easy to do –

$$\begin{aligned} d(df) &= d\left(\frac{\partial f}{\partial x_j} dx^j\right) \\ &= \frac{\partial^2 f}{\partial x_i \partial x_j} dx^i \wedge dx^j \\ &= \left(\frac{\partial^2 f}{\partial x_i \partial x_j} - \frac{\partial^2 f}{\partial x_j \partial x_i}\right) \\ &= 0, \end{aligned}$$

where in line 3 we used the antisymmetry of differential form ordering and to get zero in line 4 we note that partial derivatives commute.

This result then straightforwardly generalises, using a p-form $\omega_{j_1 \dots j_p} = \omega_J = \omega dx^J$:

$$\begin{aligned} d(d\omega) &= d\left(\sum_J d(\omega_J) \wedge dx^J\right) \\ &= \sum_J d(d\omega_J) \wedge dx^J + \sum_J \sum_{i=1}^{p} (-1)^p d\omega_J \wedge dx^{j_1} \wedge \dots \wedge dx^{j_i} \wedge \wedge dx^{j_p} \\ &= 0 \end{aligned}$$

Again we have made use of differential form ordering (note that the partial derivatives form dx^{j_i} had been moved to the end) and the fact that partial derivatives commute.

Problem 16. Induced metrics and surface areas.

Let's start by describing the tangent space. By construction, M is 2-dimensional and therefore, each constructed tangent space T_pM, at any point $p \in M$, has dimension 2 as well. So we'll need two basis vectors, and these are given by the tangent vectors to the surface $F(x, y) = z$:

$$\begin{aligned} \frac{\partial f}{\partial x} &= \left(1, 0, \frac{\partial F}{\partial x}\right) \\ \frac{\partial f}{\partial y} &= \left(0, 1, \frac{\partial F}{\partial y}\right). \end{aligned}$$

where these derivatives are evaluated at the point p. The tangent vectors are clearly tangent to the surface M and so describe a tangent space. Since they are linearly independent, each tangent vector in T_pM can be written as a linear combination of those and they span the space.

We can find an expression for the surface area directly from this:

$$\left|\frac{\partial f}{\partial x} \times \frac{\partial f}{\partial y}\right| = \sqrt{1 + \left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2}.$$

(Notice that this is simply "length of the one side times length of the other side = area", which makes sense, since we're multiplying together two basis vectors which span T_pM, a 2-dimensional manifold). The area is then the integral over the domain:

$$A = \int_U \sqrt{1 + \left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2}.$$

There is another way to find the area, as pointed out in the question sheet, by using the notion of the induced metric (see appendix A in Carroll for details). We can induce a metric in $\mathbb{R}^2$ by applying the pull-back of the metric $g_{\mu\nu}$ in $\mathbb{R}^3$:

$$G_{ab} = \frac{\partial x^\mu}{\partial y^a}\frac{\partial x^\nu}{\partial y^b} g_{\mu\nu}$$

(N.B.: here I'm using Greek script indices to refer to 3-dimensional spatial indices, not to the normal 4-dimensional spacetime ones. I do this because I want to use Latin script indices for the induced metric, a 2-dimensional object. But do not make a habit of this unless it is totally clear from context!). Now we calculate the Jacobian between the two coordinate systems:

$$\frac{\partial x^\mu}{\partial y^a} = \begin{pmatrix} 1 & 0 & \partial_x F \\ 0 & 1 & \partial_y F \end{pmatrix}.$$

This then gives you, recalling that $g_{\mu\nu} = (1, 1, 1,)$, after a page or two of algebra grinding,

$$G_{ab} = \begin{pmatrix} 1 + (\partial_x F)^2 & \partial_x F \partial_y F \\ \partial_x F \partial_y F & 1 + (\partial_y F)^2 \end{pmatrix}.$$

The area can then be found, since

$$A = \int_U \sqrt{\det(G_{ab})} = \int_U \sqrt{1 + \left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2}.$$

So, pick your poison. The second way is certainly longer - so much so I didn't want to type up all the intermediate steps. However the first way quickly becomes very unwieldy if you don't have nice vectors for the cross product. (Again, the important thing is to choose the method with which you have the most fun - or rather, the least amount of pain).

Incidentally, if we imagine we care about areas in arbitrary dimensions, the pullback method is of course the way to go, since cross products aren't defined in dimensions other than 3 or 7.[2]

[2] For those so inclined, see Hurwitz's Theorem (very heavy maths, though).

Problem 17. Topological flatness

We want to find metric which is induced by

$$ds^2 = \frac{1}{1-z^2}dx^2 + \frac{1}{1-z^2}dy^2 + \frac{1-2z^2}{(1-z^2)^2}dz^2.$$

A quick way is to simply use the map provided,

$$\begin{aligned} x &= \sin\theta\cos\phi \\ y &= \sin\theta\sin\phi \\ z &= \cos\theta \end{aligned}$$

and the associated differential elements, dx, dy, dz, and substitute these into the above metric. After some algebra, you find

$$ds^2 = d\theta^2 + d\phi^2.$$

This is actually the Minkowski metric. This means the induced manifold is flat. That is to say, the interior angles of a triangle sum to 180^o.

One can also use the pullback method.

$$G_{ab} = \frac{\partial x^\mu}{\partial y^a}\frac{\partial x^\nu}{\partial y^b}g_{\mu\nu},$$

$$\frac{\partial x^\mu}{\partial y^a} = \begin{pmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\theta\sin\phi & \sin\theta\cos\phi & 0 \end{pmatrix}.$$

This means, for instance,

$$\begin{aligned} G_{11} &= \frac{\partial x^\mu}{\partial y^1}\frac{\partial x^\nu}{\partial y^1}g_{\mu\nu} \\ &= \frac{\partial x}{\partial\theta}\frac{\partial x}{\partial\theta}g_{11} + \frac{\partial y}{\partial\theta}\frac{\partial y}{\partial\theta}g_{22} + \frac{\partial z}{\partial\theta}\frac{\partial z}{\partial\theta}g_{33} \\ &= (\cos\theta\cos\phi)^2\frac{1}{1-\cos^2\theta} + (\cos\theta\sin\phi)^2\frac{1}{1-\cos^2\theta} - \sin^2\theta\frac{1-2\cos^2\theta}{(1-\cos^2\theta)^2} \\ &= \frac{\cos^2\theta}{\sin^2\theta}(\cos^2\phi + \sin^2\phi) - \frac{1-2\cos^2\theta}{1-\cos^2\theta} \\ &= \frac{\cos^2\theta}{\sin^2\theta} - \frac{1-2\cos^2\theta}{\sin^2\theta} \\ &= 1 \end{aligned}$$

The rest I leave to you. But this confirms that the term in front of $d\theta^2$ is 1, and eventually you find that

$$G_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Problem 18. Induced metric on the torus.

Like before, we can either use the notion of pullbacks or just "plug in" the map provided.

If you want to use pullbacks – the induced metric on T^2 is the pullback of the flat metric in $\mathbb{R}^4$. The induced metric is obtained by matrix multiplication:

$$(\varphi^* g)_{\mu\nu} = \frac{\partial x^\alpha}{\partial \theta^\mu}\frac{\partial x^\beta}{\partial \theta^\nu} g_{\alpha\beta}$$

and gives, eventually,

$$g_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}.$$

Therefore,

$$ds^2 = \theta_1^2 + 4\ \mathrm{d}\theta_2^2.$$

Alternatively, substitution into the flat line element $\mathrm{d}s^2 = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2 + \mathrm{d}x_4^2$ leads to the same result

$$\begin{aligned} \mathrm{d}s^2 &= \mathrm{d}\left(\cos\left(\theta_1\right)\right)^2 + \mathrm{d}\left(\sin\left(\theta_1\right)\right)^2 + \mathrm{d}\left(2\cos\left(\theta_2\right)\right)^2 + \mathrm{d}\left(2\cos\left(\theta_2\right)\right)^2 \\ &= \mathrm{d}\theta_1^2 + 4\ \mathrm{d}\theta_2^2. \end{aligned}$$

The “volume” is given by

$$V_{T^2} = \int d\theta_1 d\theta_2 \sqrt{\det(g_{ab})} = 8\pi^2.$$

The determinant is 4 and each angle ranges from 0 to 2π.