FYS4160 - General Relativity
Problem Set 6 Solutions
Spring 2024

These solutions are credited to Jake Gordin, who wrote them in the years 2020-23.

If you spot any typos, mistakes, don't hesitate to contact me at halvor.melkild@fys.uio.no. For any physics related question please use the forum at astro-discourse.uio.no.

The idea of these solutions is to give you a sense of what a 'model' answer should be, and they also elaborate on some discussions from the help sessions. I try to make them "pedagogical": i.e. hopefully comprehensive and most steps should be explained.

Problem 19. Divergence and curl in spherical coordinates.

Let's begin by writing down our map from Cartesian to polar coordinates:

$$\begin{aligned}x &= r\cos(\phi)\sin(\theta)\\ y &= r\sin(\phi)\sin(\theta)\\ z &= r\cos(\theta)\end{aligned}$$

The equations we want to reproduce are

$$\begin{aligned}\nabla\cdot\mathbf{A} &= \frac{1}{r^2}\frac{\partial\left(r^2A_r\right)}{\partial r}+\frac{1}{r\sin(\theta)}\frac{\partial\left(A_\theta in(\theta)\right)}{\partial\theta}+\frac{1}{r\sin(\theta)}\frac{\partial\phi}{\partial\phi}\\ \nabla\times\mathbf{A} &= \frac{1}{r\sin(\theta)}\left(\frac{\partial\left(A_\phi\sin(\theta)\right)}{\partial\theta}-\frac{\partial A_\theta}{\partial\phi}\right)\mathbf{r}+\frac{1}{r}\left(\frac{1}{\sin(\theta)}\frac{\partial A_r}{\partial\phi}-\frac{\partial\left(rA_\phi\right)}{\partial r}\right)\theta+\frac{1}{r}\left(\frac{\partial\left(rA_\theta\right)}{\partial r}-\frac{\partial A_r}{\partial\theta}\right).\end{aligned}$$

We follow the hint in the question and first calculate the metric in spherical coordinates. We use the coordinate transform above and calculate $ds^2=dx^2+dy^2+dz^2$. You should get are:

$$\begin{aligned}g_{11} &= 1\\ g_{22} &= r^2\\ g_{33} &= r^2\sin^2\theta\end{aligned}$$

Next we can find the Christoffel symbols. However, this takes (1) lots of time and (2) the next question has the general procedure in more detail (note that there are two ways to find them - either by using the action and finding the geodesic equation or by calculating them from the metric directly). For the remainder of the problem though I'll just look them up (work smart // not hard). They nonzero ones are

$$\Gamma^1_{ij}=\begin{pmatrix}0&0&0\\0&-r&0\\0&0&-r\sin^2\theta\end{pmatrix}$$

$$\Gamma^2_{ij}=\begin{pmatrix}0&\frac{1}{r}&0\\\frac{1}{r}&0&0\\0&0&-\sin\theta\cos\theta\end{pmatrix}$$

$$\Gamma^3_{ij}=\begin{pmatrix}0&0&\frac{1}{r}\\0&0&\cot\theta\\\frac{1}{r}&\cot\theta&0\end{pmatrix}$$

We now turn our attention to coordinate bases and orthonormal bases. We can convert between bases: if $\{\mathbf{e}_i\}$ is a coordinate basis and $\{\tilde{\mathbf{e}}_\mathbf{i}\}$ is an orthonormal basis, then

$$\tilde{\mathbf{e}}_i = \frac{\mathbf{e}_i}{\sqrt{g_{ii}}}$$

This is needed to keep the orthonormal basis vectors, well, orthonormal. Regardless of what our chosen basis is, an orthonormal basis can always be constructed in this way.

With all this machinery setup, we can now do the question: finding the divergence and curl. Let's do the divergence first. There is a nice way to derive the expression. First, in a coordinate basis, the covariant derivative can be expressed as

$$\nabla_\alpha V^\beta = \frac{\partial V^\beta}{\partial x^\alpha} + \Gamma^\beta_{\alpha\gamma} V^\gamma.$$

The divergence of the vector field $\mathbf{A}$ sums over the components, so have

$$\nabla \cdot \mathbf{A} = \nabla_i A^i = \frac{\partial A^i}{\partial x^i} + \Gamma^i_{ik} A^k.$$

Consider just the terms that will contribute A^r:

$$\begin{aligned}
\nabla_i A^r &= \frac{\partial A^r}{\partial x^i} + \Gamma^i_{ir} A^r \\
&= \partial_r A^r + \Gamma^r_{rr} A^r + \Gamma^\theta_{\theta r} A^r + \Gamma^\phi_{\phi r} A^r \\
&= \partial_r A^r + \frac{2}{r} A^r \\
&= \frac{1}{r^2} \frac{\partial \left(r^2 A_r\right)}{\partial r}.
\end{aligned}$$

The last line followed from the reverse chain rule. The factor of 2 appears because in the sum we have, say, $\Gamma^\theta_{\theta r} A^r$ and $\Gamma^\theta_{r\theta} A^r$, but the Christoffel symbol is symmetric in its lower indices so the terms are identical. The result here is first term in the divergence expression.

N.B.: the expressions we want to derive are in an orthonormal basis; remember that our expression above is in a coordinate basis. Convert appropriately. To see this in action, consider just the terms that will contribute A^ϕ:

$$\begin{aligned}
\nabla_i A^\phi &= \frac{\partial A^\phi}{\partial x^i} + \Gamma^i_{i\phi} A^\phi \\
&= \partial_\phi A^\phi + \Gamma^r_{r\phi} A^\phi + \Gamma^\theta_{\theta\phi} A^\phi + \Gamma^\phi_{\phi\phi} A^\phi \\
&= \partial_\phi A^\phi
\end{aligned}$$

This is off from the regular expression by a factor of $1/r \sin\theta$. This is because we are looking at the vector field in an orthonormal basis. We convert by writing:

$$\tilde{\mathbf{e}}_3 = \frac{\mathbf{e}_3}{\sqrt{g_{33}}} = \frac{\mathbf{e}_3}{r \sin\theta}$$

This is the missing factor. We don't have to consider this for the case of A^r because $\sqrt{g_{11}} = 1$.

Similarly, an expression for $\nabla \times \mathbf{A}$ can be derived. Recall that

$$(\nabla \times \mathbf{A})^k = \epsilon^{ijk}\nabla_i A_j = \epsilon^{ijk} g_{jl} \nabla_i A^l = \epsilon^{ijk} g_{jl} \left(\frac{\partial A^l}{\partial x^i} + \Gamma^l_{ik} A^k \right).$$

Just be extra careful with indices and conversion between bases.

Problem 20. Varying actions: the 3-sphere.

(a) Refer first to equation 3.49 in Carroll:

$$I = \frac{1}{2} \int g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} d\tau.$$

Performing the index contractions using the 3-sphere metric, we find

$$I = \frac{1}{2} \int d\tau (\dot{\psi}^2 + \sin^2 \psi \dot{\theta}^2 + \sin^2 \psi \sin^2 \theta \dot{\phi}^2).$$

First, we vary the action, which amounts to differentiating each term with respect to whatever variables there are and keeping in a δx for each differentiation. This gives

$$\delta I = \frac{1}{2} \int d\tau (2\dot{\psi}\delta\dot{\psi} + 2\sin\psi\cos\psi\delta\psi\dot{\theta}^2 + \sin^2\psi 2\dot{\theta}\delta\dot{\theta} + 2\sin\psi\cos\psi\delta\psi\sin^2\theta\dot{\phi}^2 + \\ \sin^2\psi 2\sin\theta\cos\theta\delta\theta\dot{\phi}^2 + \sin^2\psi\sin^2\theta 2\dot{\phi}\delta\dot{\phi}).$$

When there is more than one variable per term, use the product rule. Each term has a factor of 2, so

$$\delta I = \int d\tau (\dot{\psi}\delta\dot{\psi} + \sin\psi\cos\psi\delta\psi\dot{\theta}^2 + \sin^2\psi\dot{\theta}\delta\dot{\theta} + \sin\psi\cos\psi\delta\psi\sin^2\theta\dot{\phi}^2 + \\ \sin^2\psi\sin\theta\cos\theta\delta\theta\dot{\phi}^2 + \sin^2\psi\sin^2\theta\dot{\phi}\delta\dot{\phi}).$$

Now, let's talk a bit about varying actions. We want to vary the action with respect to all available variables, but not their derivatives. A neat trick is to recall that integration by parts is given

$$\int_b^a u dv = uv\Big|_b^a - \int_b^a v du;$$

we assume the terms vanish on the boundary, so the terms when evaluated there vanish. Therefore

$$\int_b^a u dv = - \int_b^a v du.$$

We *do not* want variations w.r.t to $\delta\dot{x}$. Consider this for just the first term in the varied action; it is a $\delta\dot{\psi}$ term, which we won't want. Integration by parts lets us say that the integral of this, w.r.t to $d\tau$, can be written as

$$\int \dot{\psi}\delta\dot{\psi} = - \int \ddot{\psi}\delta\psi.$$

We can now group with this with all the other terms containing just a $\delta\psi$.

Doing this for the whole variation, we find three sets of terms: each with variations with respect to a different variable. These variations must all vanish independently, by the principle of least action, $\delta I = 0$. And so we get

$$\begin{aligned}\ddot{\psi} - \sin\psi\cos\psi\dot{\theta}^2 - \sin\psi\cos\psi\sin^2\theta\dot{\phi}^2 &= 0\\ \ddot{\theta} + 2\frac{\cos\psi}{\sin\psi}\dot{\theta}\dot{\psi} - \sin\theta\cos\theta\dot{\phi}^2 &= 0\\ \ddot{\phi} + 2\frac{\cos\psi}{\sin\psi}\dot{\psi}\dot{\phi} + 2\frac{\cos\theta}{\sin\theta}\dot{\theta}\dot{\phi} &= 0\end{aligned}$$

These are clearly all geodesic equations, of the form $\ddot{x}^\mu + \Gamma^\mu_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = 0$. We can identify the nonzero Christoffel symbols:

$$\begin{array}{llll}\Gamma^1_{22} = -\cos(\psi)\sin(\psi) & \Gamma^1_{22} = -\cos(\psi)\sin(\psi) & \Gamma^1_{33} = -\cos(\psi)\sin(\psi)\sin^2(\theta) & \\ \Gamma^2_{12} = \cot(\psi) & \Gamma^2_{21} = \cot(\psi) & \Gamma^2_{33} = -\cos(\theta)\sin(\theta) & \\ \Gamma^3_{13} = \cot(\psi) & \Gamma^3_{31} = \cot(\psi) & \Gamma^3_{32} = \cot(\theta) & \Gamma^3_{23} = \cot(\theta).\end{array}$$

(b) With the Christoffel symbols in hand, all that remains is to plug n' chug to compute the Riemann and Ricci tensor as well as the Ricci scalar. I won't include the details (sorry/not sorry) but here are the confirmed final results you can use to check if you've done it correctly:

$$\begin{array}{llll}R^1_{212} = \sin^2\psi & R^1_{221} = -\sin^2\psi & R^1_{313} = \sin^2\theta\sin^2\psi & R^1_{331} = -\sin^2\theta\sin^2\psi\\ R^2_{121} = 1 & R^2_{112} = -1 & R^2_{323} = \sin^2\theta\sin^2\psi & R^2_{332} = -\sin^2\theta\sin^2\psi\\ R^3_{131} = 1 & R^3_{113} = -1 & R^3_{232} = \sin^2\psi & R^3_{223} = -\sin^2\psi\end{array}$$

$$\begin{aligned}&R_{11} = 2\\ &R_{22} = 2\sin^2\psi\\ &R_{33} = 2\sin^2\psi\sin^2\theta\\ &R = 6.\end{aligned}$$

Remember that the Ricci tensor is a contraction of the Riemann tensor with the metric, and the Ricci scalar a contraction of the Ricci tensor.

Although I've omitted the details here, it's very important that you, for yourself, start with the Christoffel symbol and show you can calculate the answers here. This is the kind of exercise that is simply a matter of cranking through the details. It's boring and easy to mess up because of a missing term or a minus sign or whatever – but you only get better at it by doing it more; there's no conceptual or sneaky trick. It's long, I know. But sometimes physics, like life, is a long, thankless task. Have fun!

Problem 21. Riemann tensor and flatness

This is done in Carroll, pg. 124-126. I'll include here an explicit demonstration that the metric components are constant everywhere.

We start by taking it for granted that the Riemann tensor vanishes $R_{\alpha\beta\gamma\delta} = 0$. Following Carroll, this implies that the parallel transport of vectors is independent of the path and we can uniquely define a vector field $V_\alpha(x)$ just by specifying its value at one point x. The equation for parallel transport reads

$$\partial_\gamma V_\alpha = \Gamma^\sigma_{\alpha\gamma}V_\sigma.$$

We take V_α to be the gradient of a scalar Φ, then

$$\partial_\gamma \partial_\alpha \Phi = \Gamma^\sigma_{\alpha\gamma} \partial_\sigma \Phi.$$

Since the Christoffel symbols are symmetric (i.e. $\Gamma^\sigma_{\alpha\gamma} = \Gamma^\sigma_{\gamma\alpha}$), this equation is path-independent as well. We choose four independent scalar solutions $\Phi^1, \ldots, \Phi^4$ to this equation and define a new set of coordinates $\tilde{x}^\rho = \Phi^\rho$. We can write the old metric in terms of the new one in the usual way:

$$g_{\alpha\beta} = \frac{\partial \tilde{x}^\mu}{\partial x^\alpha} \frac{\partial \tilde{x}^\nu}{\partial x^\beta} \tilde{g}_{\mu\nu}$$

Note that the new coordinates are solutions to the parallel transport equation:

$$\partial_\nu \partial_\mu \tilde{x}^\rho = \Gamma^\rho_{\mu\nu} \partial_\sigma \tilde{x}^\rho,$$

with $\partial_\sigma \tilde{x}^\rho = \frac{\partial \tilde{x}^\rho}{\partial x^\sigma}$. Finally, we take the x^γ derivative on both sides of metric transformation equation:

$$\partial_\gamma g_{\alpha\beta} = \partial_\alpha \tilde{x}^\mu \partial_\beta \tilde{x}^\nu \frac{\partial \tilde{g}_{\mu\nu}}{\partial x^\gamma} + \left(\partial_\gamma \partial_\alpha \tilde{x}^\mu \partial_\beta \tilde{x}^\nu + \partial_\alpha \tilde{x}^\mu \partial_\gamma \partial_\beta \tilde{x}^\nu\right) \tilde{g}_{\mu\nu}.$$

Using our parallel transport equation, in new coordinates, we see that

$$\begin{aligned}
\partial_\gamma g_{\alpha\beta} &= \partial_\alpha \tilde{x}^\mu \partial_\beta \tilde{x}^\nu \frac{\partial \tilde{g}_{\mu\nu}}{\partial x^\gamma} + \left(\Gamma^\sigma_{\alpha\gamma} \partial_\sigma \tilde{x}^\mu \partial_\beta \tilde{x}^\nu + \Gamma^\sigma_{\beta\gamma} \partial_\alpha \tilde{x}^\mu \partial_\sigma \tilde{x}^\nu\right) \tilde{g}_{\mu\nu} \\
&= \partial_\alpha \tilde{x}^\mu \partial_\beta \tilde{x}^\nu \frac{\partial \tilde{g}_{\mu\nu}}{\partial x^\gamma} + \Gamma^\sigma_{\alpha\gamma} g_{\sigma\beta} + \Gamma^\sigma_{\beta\gamma} g_{\alpha\sigma} \\
&= \partial_\alpha \tilde{x}^\mu \partial_\beta \tilde{x}^\nu \frac{\partial \tilde{g}_{\mu\nu}}{\partial x^\gamma} + \partial_\gamma g_{\alpha\beta}
\end{aligned}$$

where the third line is the relation between the derivative of the metric components and the Christoffel. Thus, we have

$$\partial_\alpha \tilde{x}^\mu \partial_\beta \tilde{x}^\nu \frac{\partial \tilde{g}_{\mu\nu}}{\partial x^\gamma} = 0$$

which tells us (assuming that the transformation is invertible) that

$$\partial_\gamma \tilde{g}_{\mu\nu} = \frac{\partial \tilde{g}_{\mu\nu}}{\partial \tilde{x}^\gamma} = 0$$

We see that the metric is constant, in a coordinate system, and we conclude that spacetime is flat *in all coordinate systems.*

Problem 22. Deriving the Riemann curvature tensor from parallel transport.

They say life is about the journey, not the destination. This is especially true in curved spacetime.

Two vectors, V_a and V_b both end up at point y but via different paths. The difference in those vectors can be described by the Riemann curvature tensor; the expression you're supposed to derive is called the equation of geodesic deviation. There are a few ways to derive the equation. The way described here is through that of parallel transport.

There are a few nice references on the web that do this. This YouTube video is good: https://www.youtube.com/watch?v=3ULtOIRkqPU&ab_channel=TensorCalculus-RobertDavie; it is also covered in Schutz, pg. 157.