FYS4160 - General Relativity
Problem Set 7 Solutions
Spring 2024

These solutions are credited to Jake Gordin, who wrote them in the years 2020-23.

If you spot any typos, mistakes, don't hesitate to contact me at halvor.melkild@fys.uio.no. For any physics related question please use the forum at astro-discourse.uio.no.

The idea of these solutions is to give you a sense of what a 'model' answer should be, and they also elaborate on some discussions from the help sessions. I try to make them "pedagogical": i.e. hopefully comprehensive and most steps should be explained.

Problem 23. Curved d'Alembertians.

We start with the definition of $\Box$ in an arbitrary curved spacetime:

$$\Box = g_{\mu\nu}\nabla_\mu\nabla_\nu.$$

Here's a cute little trick: the Christoffel symbol can be written as[1]

$$\Gamma^\mu_{\mu\nu} = \frac{1}{\sqrt{g}}\partial_\nu(\sqrt{g}).$$

The divergence of a vector field can therefore be written as

$$\nabla_\mu V^\mu = \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}\,V^\mu).$$

Back to our question: note that f is a *scalar* test function, and therefore the quantity $\nabla^\mu f$ is a vector field. Now, we need to put them together,

$$\begin{aligned}\Box f &= g_{\mu\nu}\nabla_\mu\nabla_\nu f\\ \Box f &= \nabla_\mu\nabla^\mu f\\ \Box f &= \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}\,\nabla^\mu f).\end{aligned}$$

Problem 24. Fluid equations.

(a) Done step-by-step in Carroll, pg. 35-37, so I won't repeat the details. (P.S. I know I say this occasionally as a solution, but it isn't a valid answer in an exam lol).

(b) Here we consider the derivation of the Navier-Stokes equations[2]. This can be done by adding a term $\epsilon^{\mu\nu}$ to the energy-momentum tensor:

[1] I derive this at the end of the solutions, for those who are interested.

[2] This is entirely unrelated to physics, but I first heard of the Navier-Stokes equations in the book *Fashionable Nonsense* by Alan Sokal and Jean Bricmont. The Navier-Stokes equations come up because of a postmodern philosopher, Luce Irigaray, who claimed that they remain unsolved because they concern fluids and flow: these concepts are "feminine", in Irigaray's view, and the mostly male community of physicists are thus ill-equipped to understand such physics. Does the fact that the physics community is mostly male - now, in the 2020s, let alone the 90s - present significant challenges for women and impede the progress of physics as a whole? Absolutely. Does it mean that certain aspects of physics itself are gendered? I'll leave that as an exercise to the reader.

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + p\eta^{\mu\nu} + \epsilon^{\mu\nu}$$

Let's think carefully about what the components of the energy-momentum tensor actually represent. Looking at our handy diagram below, we can see what terms actually correspond to shear stress: *the off-diagonal, spatial components*. Therefore $\epsilon^{\mu\nu} \neq 0$ if $\epsilon^{\mu\nu} = \epsilon^{jk}$ and $j \neq k$.

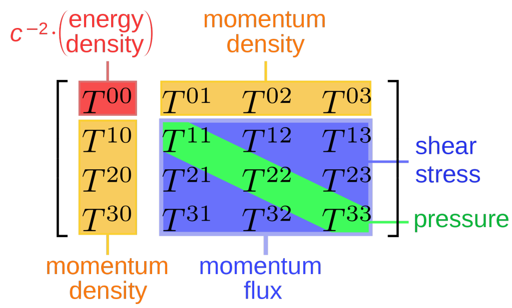

Figure 1: The breakdown of $T^{\mu\nu}$. (Taken once again from Wikipedia; if it ain't broke, don't fix it).

Recall that to derive the Euler equation, we considered the divergence of the energy-momentum tensor projected orthogonally to the fluid velocity.

$$P^\sigma{}_\nu \partial_\mu T^{\mu\nu} = 0.$$

The derivation follows the same procedure as for part (a), with the exception of an extra shear term:

$$\begin{aligned}(\delta^\sigma{}_\nu + u^\sigma u_\nu)\partial_\mu \epsilon^{\mu\nu} &= \delta^\sigma{}_\nu \partial_\mu \epsilon^{\mu\nu} + u^\sigma u_\nu \partial_\mu \epsilon^{\mu\nu} \\ &= \partial_\mu \epsilon^{\mu\sigma} + u^\sigma u_\nu \partial_\mu \epsilon^{\mu\nu}.\end{aligned}$$

As part of the Euler derivation, the choice would be made to set $\sigma = j$, and so

$$(\delta^\sigma{}_\nu + u^\sigma u_\nu)\partial_\mu \epsilon^{\mu\nu} = \partial_\mu \epsilon^{\mu j} + u^j u_\nu \partial_\mu \epsilon^{\mu\nu}.$$

We now take the non-relativistic limit (eq. 1.119 in Carroll), and sum:

$$\begin{aligned}\partial_\mu \epsilon^{\mu j} + u^j u_\nu \partial_\mu \epsilon^{\mu\nu} &= \partial_0 \epsilon^{0j} + \partial_i \epsilon^{ij} + v^j u_0 \partial_\mu \epsilon^{\mu 0} + v^j v_i \partial_\mu \epsilon^{\mu i} \\ &= \partial_i \epsilon^{ij} + v^j v_i \partial_k \epsilon^{ki} \\ &= \partial_i \epsilon^{ij} + \mathcal{O}(v^2)\end{aligned}$$

where any shear terms with a 0 component vanish because those components of $\epsilon^{\mu\nu}$ are zero. In the non-relativistic limit, also, velocities are small, which is why we neglect v^2 terms. If we define $\epsilon = -\tau$, the shear contribution to the now modified Euler equation is therefore

$$\rho(\partial_t \mathbf{v} + (\nabla \cdot \mathbf{v})) = -\nabla p + \nabla \cdot \tau$$

(in vector notation). The continuity equation does not get modified because this kind of shear, *viscous stress*, does not act parallel (or along) the fluid velocity.

Appendix: derivation of the Christoffel in terms of the metric determinant.

For some generic matrix A, we have Jacobi's formula,:

$$\frac{d}{dt}\log\det(A(t)) = \mathrm{tr}\left(A^{-1}\frac{d}{dt}A(t)\right)$$

Differentiation of the LHS gives

$$\frac{1}{\det A(t)}\frac{d}{dt}(\det(A(t))) = \mathrm{tr}\left(A^{-1}\frac{d}{dt}A(t)\right)$$

Let's consider a 4x4 matric $g_{\mu\nu}$ and use the shorthand for its determinant g; we differentiate

$$\partial_v g = (g)\,\mathrm{tr}\left(g^{-1}\partial_v g\right) = (g)g^{\mu\lambda}\partial_\nu g_{\mu\lambda}$$

Hence, the derivative of $\sqrt{g}$ would then be

$$\begin{aligned}\partial_v\sqrt{g} &= \frac{1}{2}g^{-1/2}\partial_\nu g \\ &= \frac{1}{2}g^{-1/2}gg^{\mu\lambda}\partial_\nu g_{\mu\lambda} \\ &= \frac{1}{2}\sqrt{g}g^{\mu\lambda}\partial_\nu g_{\mu\lambda}\end{aligned}$$

Dividing by $\sqrt{g}$ gives

$$\Gamma^{\mu}_{\mu\nu} = \frac{1}{2}g^{\mu\lambda}\partial_\nu g_{\mu\lambda},$$

the canonical Christoffel. (Compare to a contraction of equation 3.1 in Carroll, noting here that two indices are contracted).

Hence we can say that

$$\Gamma^{\mu}_{\mu\nu} = \frac{1}{\sqrt{\det g}}\partial_\nu(\sqrt{\det g}).$$