FYS4160 - General Relativity
Problem Set 8 Solutions
Spring 2024

These solutions are credited to Jake Gordin, who wrote them in the years 2020-23.

If you spot any typos, mistakes, don't hesitate to contact me at halvor.melkild@fys.uio.no. For any physics related question please use the forum at astro-discourse.uio.no.

The idea of these solutions is to give you a sense of what a 'model' answer should be, and they also elaborate on some discussions from the help sessions. I try to make them "pedagogical": i.e. hopefully comprehensive and most steps should be explained.

Problem 26. Some fun aspects of the Einstein-Hilbert action.

(a) The problem is essentially asking the following:

- Why, in general, do we want our actions to depend on only fields and their first derivatives? (In this case our field is the metric, $g_{\mu\nu}$).
- Show then that the Einstein-Hilbert action only, in fact, depends on the metric and its first derivative.

Re: the first part, the short answer is provided by what's called *Ostrogradski's theorem* (proved way back in 1850, before modern theoretical physics!). The theorem states:

If a non-degenerate Lagrangian, $\mathcal{L}(q, ..., q(n))$, depends on the nth derivative of a single configuration variable q, with $n > 1$, then the energy function in the corresponding Hamiltonian picture is unbounded from below.

Let's unpack this in clearer language. Basically, if a Lagrangian depends on anything with derivatives higher than $\dot{q}$, say $\ddot{q}$, $\dddot{q}$, etc, then the Hamiltonian – which gives the energy of the system – can be any value below zero, which is nonphysical. This is why we reject out of principle theories which have actions with second field derivatives or higher. They are said suffer from the so-called Ostrogradski instability. See the appendix for a simple demonstration of this theorem.

Now, we return to the Einstein-Hilbert action. Why isn't the Ostrogradski instability an issue? We can show that it isn't by expanding the action:

$$
\begin{aligned}
S_H &= \int d^4x\sqrt{-g}R \\
&= \int d^4x\sqrt{-g}R_{\mu\nu}g^{\mu\nu} \\
&= \int d^4x\sqrt{-g}R^{\lambda}{}_{\mu\lambda\nu}g^{\mu\nu}.
\end{aligned}
$$

We can see that the action contains second derivatives of the metric through the Riemann tensor. Let's look at these terms:

$$
\begin{aligned}
R^{\lambda}_{\mu\lambda\nu} &= \partial_\lambda\Gamma^{\lambda}_{\mu\nu} - \partial_\nu\Gamma^{\lambda}_{\lambda\mu} + \text{1st order derivatives} \\
&= \frac{1}{2}\partial_\lambda\left(g^{\lambda\rho}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right)\right) - \frac{1}{2}\partial_\nu\left(g^{\lambda\rho}\left(\partial_\mu g_{\lambda\rho}\right)\right) + \text{ 1st-order derivatives} \\
&= \frac{1}{2}g^{\lambda\rho}\left(\partial_\lambda\partial_\mu g_{\nu\rho} + \partial_\lambda\partial_\nu g_{\rho\mu} - \partial_\lambda\partial_\rho g_{\mu\nu} - \partial_\mu\partial_\nu g_{\lambda\rho}\right) + \text{ 1st-order derivatives}
\end{aligned}
$$

Now putting this back in the action yields

$$S_H = \int d^4x\sqrt{-g}\frac{1}{2}g^{\mu\nu}g^{\lambda\rho}\left(\partial_\lambda\partial_\mu g_{\nu\rho} + \partial_\lambda\partial_\nu g_{\rho\mu} - \partial_\lambda\partial_\rho g_{\mu\nu} - \partial_\mu\partial_\nu g_{\lambda\rho}\right) + \text{1st-order derivatives.}$$

The second derivative terms can be cast into terms containing only first derivatives using the product rule. For example, the first term is

$$\begin{aligned}\int d^4x\sqrt{-g}\frac{1}{2}g^{\mu\nu}g^{\lambda\rho}\partial_\lambda\partial_\mu g_{\nu\rho} &= \int d^4x\partial_\lambda\left(\sqrt{-g}\frac{1}{2}g^{\mu\nu}g^{\lambda\rho}\partial_\mu g_{\nu\rho}\right) \\ &\quad - \int d^4x\partial_\lambda\left(\sqrt{-g}\frac{1}{2}g^{\mu\nu}g^{\lambda\rho}\right)\partial_\mu g_{\nu\rho}.\end{aligned}$$

The first term above doesn't contribute to the equations of motion because it's a total divergence term – we assume the integral vanishes on the boundaries – and we are left with terms only containing first derivatives of the metric. Therefore the Einstein-Hilbert action only depends on the metric and its first derivatives, and thus has a well-defined energy spectrum.

(b) There are several ways to do this. I'll look at two.

Method 1: bash the door down

We recall the transformation rule for the Christoffel symbol (eq. (3.10) in Carroll):

$$\Gamma^{\lambda'}_{\mu'\nu'} = \frac{\partial y^{\lambda'}}{\partial x^\lambda}\frac{\partial x^\mu}{\partial y^{\mu'}}\frac{\partial x^\nu}{\partial y^{\nu'}}\Gamma^\lambda_{\mu\nu} + \frac{\partial y^{\lambda'}}{\partial x^\lambda}\frac{\partial^2 x^\lambda}{\partial y^{\mu'}\partial y^{\nu'}}.$$

The variation of the Christoffel symbol can be defined as

$$\delta\Gamma^{\lambda'}_{\mu'\nu'} = \Gamma^{\lambda'}_{\mu'\nu'} - \hat{\Gamma}^{\lambda'}_{\mu'\nu'}.$$

Both Christoffels transform the same way, and so

$$\begin{aligned}\delta\Gamma^{\lambda'}_{\mu'\nu'} &= \frac{\partial y^{\lambda'}}{\partial x^\lambda}\frac{\partial x^\mu}{\partial y^{\mu'}}\frac{\partial x^\nu}{\partial y^{\nu'}}\Gamma^\lambda_{\mu\nu} + \frac{\partial y^{\lambda'}}{\partial x^\lambda}\frac{\partial^2 x^\lambda}{\partial y^{\mu'}\partial y^{\nu'}} - \frac{\partial y^{\lambda'}}{\partial x^\lambda}\frac{\partial x^\mu}{\partial y^{\mu'}}\frac{\partial x^\nu}{\partial y^{\nu'}}\hat{\Gamma}^\lambda_{\mu\nu} - \frac{\partial y^{\lambda'}}{\partial x^\lambda}\frac{\partial^2 x^\lambda}{\partial y^{\mu'}\partial y^{\nu'}} \\ &= \frac{\partial y^{\lambda'}}{\partial x^\lambda}\frac{\partial x^\mu}{\partial y^{\mu'}}\frac{\partial x^\nu}{\partial y^{\nu'}}(\Gamma^\lambda_{\mu\nu} - \hat{\Gamma}^\lambda_{\mu\nu}) \\ &= \frac{\partial y^{\lambda'}}{\partial x^\lambda}\frac{\partial x^\mu}{\partial y^{\mu'}}\frac{\partial x^\nu}{\partial y^{\nu'}}\delta\Gamma^\lambda_{\mu\nu}.\end{aligned}$$

Therefore $\delta\Gamma^{\lambda'}_{\mu'\nu'}$ is a tensor.

Method 2: pick the lock

Consider the covariant derivatives associated with each Christoffel symbol in the variation

$$\delta\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \hat{\Gamma}^\lambda_{\mu\nu};$$

these would be ∇_μ and $\hat{\nabla}_\mu$. The quantity $\nabla_\mu V^\nu$ is a tensor; hence the difference $\nabla_\mu V^\nu - \hat{\nabla}_\mu V^\nu$ must also be a tensor. Using the definition of ∇_μ acting on a vector,

$$\begin{aligned}\nabla_\mu V^\nu - \hat{\nabla}_\mu V^\nu &= \partial_\mu V^\nu + \Gamma^\lambda_{\mu\lambda}V^\nu - \partial_\mu V^\nu - \hat{\Gamma}^\lambda_{\mu\lambda}v^\nu \\ &= \delta\Gamma^\lambda_{\mu\lambda}V^\nu.\end{aligned}$$

This quantity is a tensor; since V^ν alone is one too, so $\delta\Gamma^\lambda_{\mu\lambda}$ must be one as well.

Problem 27. Energy-momentum conservation for a scalar field.

Let us define

$$\hat{\mathcal{L}} = \mathcal{L}/\sqrt{-g} = -\frac{1}{2}\left(\nabla^\mu \phi\right)\left(\nabla_\mu \phi\right) - V(\phi)$$

The stress-energy tensor is then found by using equation (4.79) in Carroll (the derivation from (4.77) -(4.79) is useful here; remember the variation of the action w.r.t. the inverse metric is *defined* as the stress-energy tensor):

$$T^{\mu\nu} = \nabla^\mu \phi \nabla^\nu \phi + g^{\mu\nu}\hat{\mathcal{L}}.$$

Before we begin showing conservation, note the following:

$$\nabla^\mu \phi = \partial^\mu \phi \text{ and } \nabla^\mu V(\phi) = \partial^\mu V(\phi).$$

This is true because they are scalar quantities. We also need – and this is by no means an obvious thing to consider – that varying the Lagrangian w.r.t to the metric gives the energy-stress tensor, but varying it w.r.t the scalar field gives an equation of motion for the scalar field. You should do this yourself, it's good practice, but I'll spare you the trouble here and write down the answer:

$$\nabla^\mu \nabla_\mu \phi = \frac{dV(\phi)}{d\phi}.$$

Also remember that $\nabla_\rho g^{\mu\nu} = 0$, because of metric compatibility.

Now, we find the covariant derivative of $T^{\mu\nu}$:

$$\begin{aligned}
\nabla_\mu T^{\mu\nu} &= \nabla_\mu\left(\nabla^\mu \phi \nabla^\nu \phi\right) - g^{\mu\nu}\nabla_\mu\left(\frac{1}{2}\left(\nabla^\rho \phi\right)\left(\nabla_\rho \phi\right) + V(\phi)\right) \\
&= \nabla^\nu \phi \nabla_\mu \nabla^\mu \phi + \nabla^\mu \phi \nabla_\mu \nabla^\nu \phi - g^{\mu\nu}\nabla_\mu\left(\frac{1}{2}\left(\nabla^\rho \phi\right)\left(\nabla_\rho \phi\right)\right) - \nabla^\nu V(\phi)
\end{aligned}$$

Consider the last term. We use the chain rule to write

$$\nabla^\nu V(\phi) = \partial^\nu V(\phi) = \frac{\partial V(\phi)}{\partial \phi}\partial^\nu \phi.$$

Using the equation of motion for the scalar field, and writing the partial derivaitve of the scalar field as a covariant one, this can be written as

$$\frac{\partial V(\phi)}{\partial \phi}\partial^\nu \phi = \nabla^\mu \nabla_\mu \phi \nabla^\nu \phi.$$

This cancels the first term in the covariant derivative of $T^{\mu\nu}$. We now have

$$\begin{aligned}
\nabla_\mu T^{\mu\nu} &= \nabla^\mu \phi \nabla_\mu \nabla^\nu \phi - g^{\mu\nu}\nabla_\mu\left(\frac{1}{2}\left(\nabla^\rho \phi\right)\left(\nabla_\rho \phi\right)\right) \\
&= \nabla^\mu \phi \nabla_\mu \nabla^\nu \phi - \nabla^\nu\left(\frac{1}{2}\left(\nabla^\rho \phi\right)\left(\nabla_\rho \phi\right)\right) \\
&= \nabla^\mu \phi \nabla_\mu \nabla^\nu \phi - \frac{1}{2}\nabla_\rho \phi \nabla^\nu \nabla^\rho \phi - \frac{1}{2}\nabla^\rho \phi \nabla^\nu \nabla_\rho \phi \\
&= \frac{1}{2}\nabla^\mu \phi \nabla_\mu \nabla^\nu \phi - \frac{1}{2}\nabla_\mu \phi \nabla^\nu \nabla^\mu \phi.
\end{aligned}$$

I relabelled indices in the third line from $\rho \to \mu$. Now, the covariant derivative is torsion free, $\nabla_{[\mu}\nabla^{\nu]}\phi = 0$, so the order of the covariant derivatives doesn't matter when there are two of them, so the last line can be written as

$$\nabla_\mu T^{\mu\nu} = \frac{1}{2}\nabla^\mu\phi\nabla_\mu\nabla^\nu\phi - \frac{1}{2}\nabla_\mu\phi\nabla^\mu\nabla^\nu\phi.$$

Finally, we can, in the first term, lower the one μ and raise the other; this gives

$$\nabla_\mu T^{\mu\nu} = 0.$$

Problem 28. Kepler's problem and the orbit equation.

This problem is done in many classical physics textbooks. For example, https://www.lehman.edu/faculty/anchordoqui/chapter25.pdf. Refer to equation (25.3.8) up until (25.3.24) (and appendix A in the pdf too, for information about the integral).

Appendix: Ostrogradski instability example

As a simple illustration of the theorem, consider two theories: one which depends on $\mathcal{L}(q, \dot{q})$ and a second which depends on $\mathcal{L}(q, \dot{q}, \ddot{q})$. The Euler-Lagrange equations for the first theory look like

$$\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} = 0$$

We now express this in the Hamiltonian formalism by defining two canonical coordinates,

$$Q := q \quad P := \frac{\partial \mathcal{L}}{\partial \dot{q}}$$

and a Hamiltonian function,

$$\mathcal{H} := P\dot{q} - \mathcal{L},$$

that satisfies Hamilton's equations.

Consider now the second theory. This has the following equations of motion,

$$\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} + \frac{d^2}{dt^2}\frac{\partial \mathcal{L}}{\partial \ddot{q}} = 0,$$

canonical coordinates,

$$Q_1 := q \quad Q_2 := \dot{q} \quad P_1 := \frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \ddot{q}} \quad P_2 := \frac{\partial \mathcal{L}}{\partial \ddot{q}},$$

and Hamiltonian,

$$\mathcal{H} := P_1\dot{q} + P_2\ddot{q} - \mathcal{L}.$$

Both Hamiltonians depend linearly on the canonical momenta ($\mathcal{H} \propto P$). Since the momenta can take arbitrary negative values, it seems that both Hamiltonians will be unbounded from below. But, there is one crucial difference.

In the statement of the theorem, the Lagrangian was specified to be non-degenerate. This is a rather technical condition that means the so-called Hessian matrix is invertible,

$$det\left[\frac{\partial^2 \mathcal{L}}{\partial x^2}\right] \neq 0$$

where the coordinate x is the highest order derivative coordinate, so $\dot{q}$ in the first case and $\ddot{q}$ in the second. All this means is that canonical coordinates can be inverted. In the first case, $\dot{q}$ can be rewritten as a function of P and Q, and the first Hamiltonian takes the form:

$$\mathcal{H} = \mathrm{Pf}(P, Q) - \mathcal{L}$$

So if $f(P, Q)$ has a suitable form, the linear dependence on P can be removed and the Hamiltonian is bounded from below. In the second case, it is $\ddot{q}$ that can be rewritten as a function of Q_1, Q_2, and P_2, and the second Hamiltonian takes the form:

$$\mathcal{H} = P_1 Q_2 + P_2 f\left(Q_1, Q_2, P_2\right) - \mathcal{L}$$

As before, the linear dependence on P_2 can be removed if $f\left(Q_1, Q_2, P_2\right)$ has a suitable form, but the linear dependence on P_1 cannot be removed. Thus the higher-order Hamiltonian is unbounded from below. This is the source of the Ostrogradski instability.