FYS4160 - General Relativity
Problem Set 9 Solutions
Spring 2024

These solutions are credited to Jake Gordin, who wrote them in the years 2020-23.

If you spot any typos, mistakes, don't hesitate to contact me at halvor.melkild@fys.uio.no. For any physics related question please use the forum at astro-discourse.uio.no.

The idea of these solutions is to give you a sense of what a 'model' answer should be, and they also elaborate on some discussions from the help sessions. I try to make them "pedagogical": i.e. hopefully comprehensive and most steps should be explained.

Problem 29. A heckin' zoom in Schwarzschild geometry.

We refer first to the equation just above eq. (6.15) in Carroll:

$$a^\mu = u^\sigma \nabla_\sigma u^\mu,$$

where u^μ is the 4-velocity, $dx^\mu/d\tau$. The 4-acceleration can be calculated,

$$\begin{aligned} a^\mu &= u^\sigma \nabla_\sigma u^\mu \\ &= u^\sigma \partial_\sigma u^\mu + u^\sigma \Gamma^\mu_{\sigma\rho} u^\rho. \end{aligned}$$

Since the observer is stationary, the curve u^μ is constant. Hence the first term is zero. Additionally, only u^0 is nonzero - being stationary implies the 3-velocity is zero. We have left

$$a^\mu = (u^0)^2 \Gamma^\mu_{00}.$$

We can calculate u^0 by recalling the normalisation condition for the 4-velocity:

$$\begin{aligned} u^\mu u_\mu &= -1 \\ (u^0)^2 g_{00} &= -1 \\ \implies u^0 &= \frac{1}{\sqrt{-g_{00}}}. \end{aligned}$$

From the Schwarzschild metric, $g_{00} = -(1 - r_s/r)$. We can also use eq. (5.52) in Carroll to find the only nonzero Christoffel with subscript 00, and it's Γ^1_{00}. The 4-acceleration is therefore

$$\begin{aligned} a^\mu &= \frac{1}{\left(1 - \frac{r_s}{r}\right)} \Gamma^\mu_{00} \\ a^1 &= \frac{1}{\left(1 - \frac{r_s}{r}\right)} \frac{r_s}{2r^2} \left(1 - \frac{r_s}{r}\right) \\ a^1 &= \frac{r_s}{2r^2}. \end{aligned}$$

We identify the index 1 with r, of course, and hence the *radial* acceleration is

$$a^r = \frac{r_s}{2r^2} = \frac{GM}{r^2}.$$

This is exactly the same as the Newtonian result! However, the question wants the acceleration the observer experiences, i.e. in her frame. We need then the proper acceleration, which is given by the magnitude of the 3-acceleration, $\alpha = \sqrt{a^j a_j}$. Since only the r component is nonzero and we contract indices using the Schwarzschild metric, we have $\alpha = \sqrt{g_{rr}} a^r$:

$$\alpha = \frac{GM}{r^2} \frac{1}{\sqrt{1 - \frac{2GM}{r}}}.$$

This approaches the Newtonian result far away from the black hole, at $r \gg r_s$. As $r \to r_s$, the acceleration diverges – it would take an infinite force to accelerate away from the horizon at $r = r_s$.

Problem 30. Eddington-Finkelstein lightcones.

Consider the Schwarzschild metric in the (r,t)-plane,

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2,$$

where I've introduced the notational shorthand $f(r) = 1 - r_s/r$. Null geodesics follow trajectories with $ds^2 = 0$. This means our Schwarzschild metric becomes

$$\frac{dt}{dr} = \pm\frac{1}{f(r)}.$$

The positive solution refers to outgoing geodesics (away from the black hole); the negative solution to ingoing ones. We can integrate this to solve for t. Doing so, and introducing the tortoise coordinate, $r_\star = r + r_s \ln(r/r_s - 1)$, we find,

$$t = \pm r_\star + \text{const.}$$

where we also have a constant of integration. In Eddington-Finkelstein coordinates, $v = t + r_\star$ and $u = t - r_\star$. Hence, we deduce that $u = \text{const.}$ characterises the outgoing null geodesics and $v = \text{const.}$ characterises the ingoing null geodesics. Refer to Carroll, figs. 5.7, 5.10, 5.11, for a discussion of the light cones. Notably, the lightcones don't close up in EF coordinates.

Problem 31. Dropping a beacon in a black hole.

Problem 5.5 in Carroll reads:

Consider a comoving observer sitting at constant spatial coordinates (r_*, θ_*, ϕ_*), around a Schwarzschild black hole of mass M. The observer drops a beacon into the black hole (straight down, along a radial trajectory). The beacon emits radiation at a constant wavelength $\lambda_{\rm em}$ (in the beacon rest frame).

(a) Calculate the coordinate speed dr/dt of the beacon, as a function of r.
(b) Calculate the proper speed of the beacon. That is, imagine there is a comoving observer at fixed r, with a locally inertial ccordinate system set up as the beacon passes by, and calculate the speed as measured by the comoving observer. What is it at $r = 2GM$?
(c) Calculate the wavelength λ_{obs}, measured by the observer at r_*, as a function of the radius $r_{\rm em}$ at which the radiation was emitted.
(d) Calculate the time $t_{\rm obs}$ at which a beam emitted by the beacon at radius $r_{\rm em}$ will be observed at r_*.
(e) Show that at late times, the redshift grows exponentially: $\lambda_{\rm obs}/\lambda_{\rm em} \propto e^{t_{\rm obs}/T}$. Give an expression for the time constant T in terms of the black hole mass M.

(a) We want the coordinate speed, dr/dt, of the beacon in the observer frame. To do so, we need to find a way to relate the proper speed to the coordinate speed. This is achieved through the chain rule,

$$\frac{dr}{dt} = \frac{dr}{d\tau}\frac{d\tau}{dt}.$$

We can find $dr/d\tau$ and $dt/d\tau$ using the line element, but not independently (we want something like $dt/d\tau = (stuff)$, without the other variable involved). To find them independently, we make use of conserved quantities. Recall that the energy associated with the Killing vector in Schwarzschild coordinates is

$$E = f(r)\frac{dt}{d\tau} \qquad \text{(Carroll eq. (5.61)).}$$

This Killing vector is timelike; we can get the line element into this form by considering the point $r = r_\star$. Recall the beacon starts at this point and then emits light at r_{em}. At $r = r_\star$, $dr/d\tau = 0$, since the beacon is stationary just as its thrown. Hence the we have

$$\frac{dt}{d\tau}\Big|_{r=r_\star} = \frac{1}{\sqrt{f(r_\star)}}.$$

We identify this as the energy, $E(r_\star) = \sqrt{f(r_\star)}$. However, this quantity is conserved, so is true for all values of r;

$$E = f(r)\frac{dt}{d\tau} = \sqrt{f(r_\star)}.$$

This gives us $dt/d\tau$. Now we can write the line element as

$$d\tau^2 = -ds^2 = f(r)dt^2 - \frac{1}{f(r)}dr^2.$$

Dividing through by $d\tau^2$,

$$\begin{aligned} 1 &= f(r)\left(\frac{dt}{d\tau}\right)^2 - \frac{1}{f(r)}\left(\frac{dr}{d\tau}\right)^2 \\ 1 &= \frac{E^2}{f(r)} - \frac{1}{f(r)}\left(\frac{dr}{d\tau}\right)^2 \\ 1 &= \frac{f(r_\star)}{f(r)} - \frac{1}{f(r)}\left(\frac{dr}{d\tau}\right)^2. \end{aligned}$$

Solving for the $dr/d\tau$ we get

$$\frac{dr}{d\tau} = \pm\sqrt{\frac{r_s}{r} - \frac{r_s}{r_\star}}.$$

We can now apply the chain rule from the very beginning to get

$$\frac{dr}{dt} = -f(r)\sqrt{\frac{r_s(r_\star - r)}{r(r_\star - r_s)}}.$$

(b) A nice bit of reasoning here: the relation between proper and coordinate acceleration in the previous problem was shown to be the component scaled by the square of the metric factor. Since the $x^0 = dt$ component is scaled by $\sqrt{-f(r)}$ and the $x^1 = dr$ component is scaled by $1/\sqrt{f(r)}$, the speed (i.e. ratio of dr/dt) gives a scaling of $-1/f(r)$ which exactly cancels the factor we have from part (a). Hence the proper speed is

$$\left|\frac{dr_{obs}}{dt_{obs}}\right| = \sqrt{\frac{r_s(r_\star - r)}{r(r_\star - r_s)}}.$$

This is of course the coordinate speed in terms of the observer's proper time. Note that beacon starts at $r_\star$, and so the speed becomes $c=1$ at the horizon - while the coordinate speed vanishes.

(c) We use equation (5.100): the frequency of a photon travelling on a null geodesic $x^\mu(\lambda)$, as observed by an observer travelling with 4 -velocity u^μ is

$$\omega=-g_{\mu\nu}u^\mu\frac{dx^\nu}{d\lambda_\gamma}.$$

Let's first find the frequency in the frame of the beacon, when it is at $r=r_{em}$. It follows a radial path, and therefore its 4 -velocity is $u_b^\mu=\left(dt/d\tau_b, dr/d\tau_b, 0, 0\right)$. Hence,

$$\begin{aligned}
\omega_{em} &= -g_{\mu\nu}u_b^\mu\frac{dx^\nu}{d\lambda_\gamma}\\
&=\left(1-r_s/r_{em}\right)\frac{dt}{d\tau_b}\frac{dt}{d\lambda_\gamma}-\left(1-r_s/r\right)^{-1}\frac{dr}{d\tau_b}\frac{dr}{d\lambda_\gamma}\\
&=\left(1-r_s/r_{em}\right)\frac{dt}{d\tau_b}\frac{dt}{d\lambda_\gamma}-\frac{dr}{d\tau_b}\frac{dt}{d\lambda_\gamma}\\
&=\frac{dt}{d\lambda_\gamma}\left(\sqrt{1-r_s/r_*}+\sqrt{r_s/r_{em}-r_s/r_*}\right)\\
&=\frac{E_\gamma}{1-r_s/r_{em}}\left(\sqrt{1-r_s/r_*}+\sqrt{r_s/r_{em}-r_s/r_*}\right).
\end{aligned}$$

A lot happened in each line, let's go through it bit by bit. Firstly, $E_\gamma=\left(1-r_s/r\right)dt/d\lambda_\gamma$ is a constant of motion. In line 3 we used the fact that, for photons, which are massless,

$$\frac{dr}{d\lambda_\gamma}=\frac{dr}{dt}\frac{dt}{d\lambda_\gamma}=\left(1-r_s/r\right)\frac{dt}{d\lambda_\gamma}$$

(compare to the answer for part (a), which is for a massive object – and also understand how I got this relation – it follows from $ds^2=0$ for photons). In line 4 we used

$$\frac{dr}{d\tau}=\pm\sqrt{\frac{r_s}{r}-\frac{r_s}{r_\star}}.$$

The comoving observer is at rest at $r=r_*$,, as known from the question, so the spatial components of their 4-velocity u_{obs}^μ vanish. With the time component $dt/d\tau_{obs}=\sqrt{f(r)}$, we find that they observe the photon frequency as

$$\begin{aligned}
\omega_{obs} &= -g_{\mu\nu}u_{obs}^\mu\frac{dx^\nu}{d\lambda_\gamma}\\
&=\left(1-r_s/r_*\right)\left(1-r_s/r_*\right)^{-1/2}\frac{dt}{d\lambda_\gamma}\\
&=\frac{E_\gamma}{\sqrt{1-r_s/r_*}}
\end{aligned}$$

Therefore, using $\omega=2\pi/\lambda$, we find

$$\frac{\lambda_{obs}}{\lambda_{em}}=\frac{\omega_{em}}{\omega_{obs}}=\frac{\sqrt{1-r_s/r_*}}{1-r_s/r_{em}}\left(\sqrt{1-r_s/r_*}+\sqrt{r_s/r_{em}-r_s/r_*}\right).$$

(d) We need the time for the beacon to reach its emitting point and then the time the light takes to reach the observer from the emitting point. The coordinate time required for the beacon to reach $r = r_{em}$ from $r_\star$ can be obtained by integrating the answer to part (a):

$$t_1 = -\int_{r_*}^{r_{em}} (1 - r_s/r)^{-1} \sqrt{\frac{r\,(r_* - r_s)}{r_s\,(r_* - r)}} dr$$

This integral is quite nontrivial, so I'll leave it in this form. The coordinate time taken by the photon to reach the observer is found by integrating the coordinate time of the observer in their frame:

$$t_2 = \int_{r_{em}}^{r_*} (1 - r_s/r)^{-1}\, dr = r_s \int_{r_{em}/r_s}^{r_*/r_s} \frac{x dx}{x - 1} = r_s \int_{r_{em}/r_s}^{r_*/r_s} \left[1 + \frac{1}{x - 1}\right] dx$$

finally yielding

$$t_2 = r_* - r_{em} + r_s \ln\left(\frac{r_* - r_s}{r_{em} - r_s}\right)$$

The total coordinate time when the photon is observed is $t_{obs} = t_1 + t_2$.

(e) At late times, the infalling beacon approaches the black hole closely ($r_{em} \to r_s$) and the answer to part (c) becomes

$$\frac{\lambda_{obs}}{\lambda_{em}} \longrightarrow 2\frac{1 - r_s/r_*}{1 - r_s/r_{em}} \propto \frac{r_* - r_s}{r_{em} - r_s}$$

Note that in this limit $t_1 \to t_2$, and the log term in t_2 dominates, so that the time of observation t_{obs} becomes

$$t_{obs} \to 2t_2 \approx 2r_s \ln\frac{r_* - r_s}{r_{em} - r_s}.$$

Hence, $\lambda_{\text{obs}}/\lambda_{em} \propto e^{t_{obs}/T}$, where $T = 2r_s = 4GM$.