FYS4160 - General Relativity
Problem Set 10 Solutions
Spring 2024

These solutions are credited to Jake Gordin, who wrote them in the years 2020-23.

If you spot any typos, mistakes, don't hesitate to contact me at halvor.melkild@fys.uio.no. For any physics related question please use the forum at astro-discourse.uio.no.

The idea of these solutions is to give you a sense of what a 'model' answer should be, and they also elaborate on some discussions from the help sessions. I try to make them "pedagogical": i.e. hopefully comprehensive and most steps should be explained.

Problem 32. Particle spiralling into a black hole.

Problem 5.3 in Carroll reads:

Consider a particle (not necessarily on a geodesic) that has fallen inside the event horizon, $r < 2GM$. Use the ordinary Schwarzschild coordinates (t, r, θ, ϕ). Show that the radial coordinate must decrease at a minimum rate given by

$$\left|\frac{dr}{d\tau}\right| \geq \sqrt{\frac{2GM}{r} - 1}$$

Calculate the maximum lifetime for a particle along a trajectory from $r = 2GM$ to $r = 0$. Express this in seconds for a black hole with mass measured in solar masses. Show that this maximum proper time is achieved by falling freely with $E \to 0$.

We start with calculating the rate of radial infall. Inside r_s, we have $1 - r_s/r < 0$. We recall the 4 -velocity normalisation

$$g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -1.$$

Neglecting angular coordinates, it then follows that

$$1 = (r_s/r - 1)^{-1}\left(\frac{dr}{d\tau}\right)^2 - (r_s/r - 1)\left(\frac{dt}{d\tau}\right)^2.$$

The positive term here must be greater than or equal to 1; and so,

$$(r_s/r - 1)^{-1}\left(\frac{dr}{d\tau}\right)^2 \geq 1 \quad \Longrightarrow \quad \left|\frac{dr}{d\tau}\right| \geq \sqrt{(r_s/r - 1)}.$$

Now that we have shown this relation, we want to calculate the maximum lifetime of a particle that starts at the Schwarzschild horizon and heads towards the singularity. Since

$$\left|\frac{dr}{d\tau}\right| \geq \sqrt{\frac{2GM}{r} - 1},$$

maximal proper time would be achieved when the equality holds. This is because we want dr to be as small as possible. Integrating this, we get

$$\tau_{\max} = -\int_{r_s}^{0}\frac{dr}{\sqrt{r_s/r - 1}} = 2r_s\int_0^\infty \frac{dx}{(1 + x^2)^2} = 2r_s\int_0^{\pi/2}\cos^2(y)dy = \frac{\pi}{2}r_s = \pi GM,$$

where we have performed integration by substitution, namely with $x = \sqrt{r_s/r - 1}$ and $y = \arctan x$. One could alternatively use Wolfram alpha or your favourtie integral solver.

Next, we express this time in terms of seconds and solar masses. Check the dimensions of πGM in SI units: $[GM] = m^3/s^2$, so restoring the factors c in the expression to get just seconds, we have

$$\tau_{\text{max}} = \pi GM/c^3 \sim 4.9\pi \frac{M}{M_\odot} \mu\text{s}.$$

Finally, we now consider free fall along a geodesic. We know that $E = (1 - r_s/r)\, dt/d\tau$ is constant. With the proper time defined by $d\tau^2 = -ds^2$, we find

$$\begin{aligned} d\tau^2 &= -\left(r_s/r - 1\right) dt^2 + \left(r_s/r - 1\right)^{-1} dr^2 \\ &= -E^2 \left(r_s/r - 1\right)^{-1} dt^2 + \left(r_s/r - 1\right)^{-1} dr^2 \end{aligned}$$

Since the term with E subtracts off a quantity, the maximal proper time - i.e. making the LHS the biggest - is achieved when $E = 0$.

Problem 33. A deleted Interstellar scene.

Before we begin, it should be noted that this question is very hard for very unusual reasons: the types of algebra guessing and classical mechanics reasoning are not at all obvious, although the maths isn't "hard". Don't be discouraged by finding this problem challenging: it very much is!

(a) To escape non-radially, they'll have to fight rotational motion too. This will require more force - and hence the ejection of more mass - to escape. It is thus the more prudent course of action to escape radially.

As Carroll says, "Together the conserved quantities E and L provide a convenient way to understand the orbits of particles in the Schwarzschild geometry." Indeed. Eq. (5.64) in Carroll reads

$$-E^2 + \left(\frac{dr}{d\tau}\right)^2 + \left(1 - \frac{r_s}{r}\right)\left(\frac{L^2}{r^2} + \epsilon\right) = 0.$$

In particular, for radial motion, $L = 0$ (eq. 5.62). This gives

$$-E^2 + \left(\frac{dr}{d\tau}\right)^2 + \left(1 - \frac{r_s}{r}\right)\epsilon = 0.$$

At $r \to \infty$, $E = 1$. It must always be this value at any r, since it is a conserved quantity. Additionally, $\epsilon = 1$ for massive particles. Hence at $r = R_{\text{ship}}$,

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{r_s}{R_{\text{ship}}}.$$

The minimal 4-velocity is thus

$$u^\alpha = \left(\left(1 - \frac{r_s}{R_{\text{ship}}}\right)^{-1}, \sqrt{\frac{r_s}{R_{\text{ship}}}}, 0, 0\right).$$

where $dt/d\tau$ is from eq. (5.61).

(b) Quite a bit of setup is required for this one. We need to some symbols with which to denote stuff. Before the throw, we have the rest mass m_{rest} and 4-velocity u^α_{rest}. After the throw, we have two separated fragments, the escaping part and the portion that is ejected: m_{esc}, m_{ej}, u_{esc}, u_{ej}. We're also going to define $r_s/R_{\text{ship}} \equiv \xi$. The rest and escape velocities come from the answer to part (a), and are

$$\begin{aligned} u^\alpha_{\text{rest}} &= \left((1 - \xi)^{-1}, 0, 0, 0\right) \\ u^\alpha_{\text{esc}} &= \left((1 - \xi)^{-1}, \sqrt{\xi}, 0, 0\right). \end{aligned}$$

Let's return to good (bad) old classical mechanics – 3-momentum is conserved, so $p_i^j = p_f^j$ (the upstairs indices are the indices; downstairs just means 'initial' and ''final'). Remember, 3-momentum, not 4-momentum! This gives

$$\begin{aligned} 0 &= m_{\rm esc} u_{\rm esc}^j + m_{\rm ej} u_{\rm ej}^j \\ 0 &= m_{\rm esc} \sqrt{\xi} + m_{\rm ej} u_{\rm ej}^r \\ u_{\rm ej}^r &= -\frac{m_{\rm esc}}{m_{\rm ej}} \sqrt{\xi}. \end{aligned}$$

This gives us the radial component of the ejected 4-velocity. We use the normalisation of the 4-velocity to find the time component, $u_{\rm ej}^t$:

$$\begin{aligned} g_{\alpha\beta} u^\alpha u^\beta &= -1 \\ -(1-\xi)(u_{\rm ej}^t)^2 + -(1-\xi)^{-1}(u_{\rm ej}^r)^2 &= -1 \\ (u_{\rm ej}^t)^2 &= (1-\xi)^{-2}\left((1-\xi) + (u_{\rm ej}^r)^2\right) \\ (u_{\rm ej}^t)^2 &= (1-\xi)^{-2}\left((1-\xi) + \left(\frac{m_{\rm esc}}{m_{\rm ej}}\right)^2 \xi\right) \\ u_{\rm ej}^t &= (1-\xi)^{-1}\left(1 - \xi\left(1 - \left(\frac{m_{\rm esc}}{m_{\rm ej}}\right)^2\right)\right)^{1/2}. \end{aligned}$$

In line 3 we took terms to the RHS and took out a common factor of $(1-\xi)^{-2}$; in line 4 we used our expression for $u_{\rm ej}^r$; in line 5 we massaged some terms and took the square root. We have the ejection 4-velocity as

$$u_{\rm ej}^\alpha = \left((1-\xi)^{-1}\left(1 - \xi\left(1 - \left(\frac{m_{\rm esc}}{m_{\rm ej}}\right)^2\right)\right)^{1/2}, \frac{m_{\rm esc}}{m_{\rm ej}}\sqrt{\xi}, 0, 0\right).$$

Multiplying this by the ejected mass will give you the ejected 4-momentum.

We now consider the energy before ejection. The energy is the time component of the 4-momentum, and is given by $p_{\rm rest}^t = m_{\rm rest}(1-\xi)^{-1/2}$, where $m_{\rm rest}$ is the total mass (or rest mass). Not to get lost here: we want the biggest fraction of $m_{\rm esc}/m_{\rm rest}$. After ejection we have

$$\begin{aligned} m_{\rm rest}(1-\xi)^{-1/2} &= m_{\rm esc} u_{\rm esc}^t + m_{\rm ej} u_{\rm ej}^t \\ m_{\rm rest}(1-\xi)^{-1/2} &= m_{\rm esc}(1-\xi)^{-1} + m_{\rm ej}(1-\xi)^{-1}\left(1 - \xi\left(1 - \left(\frac{m_{\rm esc}}{m_{\rm ej}}\right)^2\right)\right)^{1/2} \\ (1-\xi)^{1/2} &= \left(\frac{m_{\rm esc}}{m_{\rm rest}}\right) + \left(\frac{m_{\rm ej}}{m_{\rm rest}}\right)\left(1 - \xi\left(1 - \left(\frac{m_{\rm esc}}{m_{\rm ej}}\right)^2\right)\right)^{1/2}. \end{aligned}$$

Note that $m_{\rm rest} \neq m_{\rm ej} + m_{\rm ej}$ – the 4-momentum is conserved but the total rest mass is not. For fixed $m_{\rm rest}$ and $\xi, m_{\rm esc}$ is just a function of m_{ej}, with $m_{\rm ej} \le m_{\rm rest}$. If you massage the alegbra, you can actually isolate a term of $m_{\rm esc}^2 = ...$ The larger root is,

$$m_{\rm esc} = \frac{\tilde{M} - \xi m_{\rm ej} + \sqrt{\xi(1-\xi)\tilde{M}^2 + \left(m_{\rm ej} - \xi\tilde{M}\right)^2}}{1-\xi} \qquad \text{where } \tilde{M} \equiv (1-\xi)^{1/2} m_{\rm rest}.$$

We now see manifestly that $m_{\rm esc}$ is maximised for $m_{\rm ej} = 0$. In the limit, $m_{\rm ej} \to 0$, the above expression becomes,

$$\begin{aligned} (1-\xi)^{1/2} &= \frac{m_{\rm esc}}{m_{\rm tot}} + \xi^{1/2}\frac{m_{\rm esc}}{m_{\rm tot}} \\ f = \frac{m_{\rm esc}}{m_{\rm tot}} &= \frac{(1-\xi)^{1/2}}{1+\xi^{1/2}} \end{aligned}$$

which means spaceship cannot escape at $R_{\text{ship}} = r_s$, since at this value the maximum faction of escaped ship is zero.

Problem 34. Black hole surface areas.

(a) The induced metric at the Schwarzschild surface is that for which dt and dr are constant. This means the metric is

$$ds^2 = r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2.$$

Note that $r = r_s$. The surface area is then given by (cf. similar problems in problem set 5)

$$\begin{aligned} A_{\text{Schwarz}} &= \int_U \sqrt{\det(G_{ab})} \\ &= \int_0^\pi d\theta \int_0^{2\pi} d\phi \left[r^2 \sin\theta\right] \\ &= 4\pi^2 r_s^2 \end{aligned}$$

This result could have been obtained instantly by noting that a Schwarzschild black hole is a sphere, which therefore has a surface area of $A = 4\pi^2 R^2$. (And following the hint, null surfaces are in fact coordinate invariant).

(b) The induced Kerr metric is also found when dt and dr are constant; we have two radius choices, however. Therefore, for $dt = const$ and $r = r_\pm$, the metric is

$$ds^2 = \left(r_\pm^2 + a^2\cos^2\theta\right) \mathrm{d}\theta^2 + \left(\frac{4m^2 r_\pm^2}{r_\pm^2 + a^2\cos^2\theta}\right) \sin^2\theta \mathrm{d}\phi^2,$$

or equivalently

$$ds^2 = \left(r_\pm^2 + a^2\cos^2\theta\right) \mathrm{d}\theta^2 + \left(\frac{\left[r_\pm^2 + a^2\right]^2}{r_\pm^2 + a^2\cos^2\theta}\right) \sin^2\theta \mathrm{d}\phi^2$$

While the horizons are topologically spherical, they are emphatically not geometrically spherical (fig. 6.7, Carroll) - which is why you can't use the same trick as in part (a). The area of the horizons is calculated straightforwardly

$$A_{\text{Kerr}}^{\pm} = 4\pi \left(r_\pm^2 + a^2\right).$$

This reduces to the Schwarzschild area as $a \to 0$, since then $r_+ = r_-$ (see eq. 6.82 in Carroll).