FYS4160 - General Relativity
Problem Set 11 Solutions
Spring 2024

These solutions are credited to Jake Gordin, who wrote them in the years 2020-23.

If you spot any typos, mistakes, don't hesitate to contact me at halvor.melkild@fys.uio.no. For any physics related question please use the forum at astro-discourse.uio.no.

The idea of these solutions is to give you a sense of what a 'model' answer should be, and they also elaborate on some discussions from the help sessions. I try to make them "pedagogical": i.e. hopefully comprehensive and most steps should be explained.

Problem 35. Linearised gravity.

(a) Consider a global Lorentz transformation of $g_{\mu\nu}$,

$$g'_{\alpha\beta} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \, g_{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta (\eta_{\mu\nu} + h_{\mu\nu}).$$

$\eta_{\mu\nu}$ is invariant by definition, so we have

$$g'_{\alpha\beta} = \eta'_{\mu\nu} + \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta h_{\mu\nu} \implies g'_{\alpha\beta} - \eta'_{\alpha\beta} = h'_{\alpha\beta} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta h_{\mu\nu}$$

and we see that $h_{\mu\nu}$ transforms as a rank-2 tensor on a Minkowski background.

Since the Lagrangian is constructed from fully contracted objects, it is Lorentz invariant (it's a scalar boi) and thus the theory is Lorentz invariant on flat spacetime.

(b) Equation (7.9) is

$$\mathcal{L} = \frac{1}{2}\left[(\partial_\mu h^{\mu v})(\partial_\nu h) - (\partial_\mu h^{\rho\sigma})(\partial_\rho h^\mu_\sigma) + \frac{1}{2}\eta^{\mu\nu}(\partial_\mu h^{\rho\sigma})(\partial_\nu h_{\rho\sigma}) - \frac{1}{2}\eta^{\mu\nu}(\partial_\mu h)(\partial_\nu h)\right].$$

This is the Lagrangian for the matter-free part. We can take out a common factor of $\eta^{\mu\nu}$ and write this as

$$\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\left[(\partial_\rho h^{\rho\sigma})(\partial_\sigma h_{\mu\nu}) - (\partial_\mu h^{\rho\sigma})(\partial_\rho h_{\nu\sigma}) + \frac{1}{2}(\partial_\mu h^{\rho\sigma})(\partial_\nu h_{\rho\sigma}) - \frac{1}{2}\eta^{\alpha\beta}\eta_{\rho\sigma}(\partial_\mu h^{\rho\sigma})(\partial_\nu h_{\alpha\beta})\right].$$

The action is given by $S = \int d^4x \mathcal{L}$. We vary this w.r.t. $h^{\rho\sigma}$ to obtain

$$\begin{aligned}\delta S = \frac{1}{2}\eta^{\mu\nu}\int d^4x \, [&(\partial_\rho \delta h^{\rho\sigma})(\partial_\sigma h_{\mu\nu}) + (\partial_\rho h^{\rho\sigma})(\partial_\sigma \delta h_{\mu\nu}) - (\partial_\mu \delta h^{\rho\sigma})(\partial_\rho h_{\nu\sigma}) - (\partial_\mu h^{\rho\sigma})(\partial_\rho \delta h_{\nu\sigma}) \\ &+ \frac{1}{2}(\partial_\mu \delta h^{\rho\sigma})(\partial_\nu h_{\rho\sigma}) + \frac{1}{2}(\partial_\mu h^{\rho\sigma})(\partial_\nu \delta h_{\rho\sigma}) - \frac{1}{2}\eta^{\alpha\beta}\eta_{\rho\sigma}(\partial_\mu \delta h^{\rho\sigma})(\partial_\nu h_{\alpha\beta}) - \frac{1}{2}\eta^{\alpha\beta}\eta_{\rho\sigma}(\partial_\mu h^{\rho\sigma})(\partial_\nu \delta h_{\alpha\beta})\Big] \\ = -\frac{1}{2}\eta^{\mu\nu}\int d^4x \, [&(\delta h^{\rho\sigma})(\partial_\rho\partial_\sigma h_{\mu\nu}) + (\partial_\sigma\partial_\rho h^{\rho\sigma})(\delta h_{\mu\nu}) - (\delta h^{\rho\sigma})(\partial_\mu\partial_\rho h_{\nu\sigma}) - (\partial_\rho\partial_\mu h^{\rho\sigma})(\delta h_{\nu\sigma}) \\ &+ (\delta h^{\rho\sigma})(\partial_\mu\partial_\nu h_{\rho\sigma}) - \eta^{\alpha\beta}\eta_{\rho\sigma}(\delta h^{\rho\sigma})(\partial_\mu\partial_\nu h_{\alpha\beta})] \,.\end{aligned}$$

This can look overwhelming, so let's break in down. In the first line 1 used the product rule and made sure to vary every metric term that appears. The second line follows from integration by parts. Recall: $udv = uv - \int vdu$. We assume the boundary term always vanishes at infinity, so we really have $udv = -\int vdu$. The amounts to swapping which term has the derivative and adding in a minus sign. For example, $(\partial_\rho h^{\rho\sigma})(\partial_\sigma \delta h_{\mu\nu})$ can be written as $\partial_\rho h^{\rho\sigma}\partial_\sigma \delta h_{\mu\nu} = -\partial_\sigma\partial_\rho h^{\rho\sigma}\delta h_{\mu\nu}$. I have taken out the overall minus sign in the second line, since I have applied IBP to every term.

We chose our variation parameter to be $\delta h^{\sigma\rho}$. It could have been $h_{\rho\sigma}$ or $h_{\mu\nu}$; the important thing is that you're consistent. I chose $\delta h^{\sigma\rho}$ since most terms already have a factor of $h^{\sigma\rho}$ - it saves us some work.

Since we chose $\delta h^{\sigma\rho}$, we want to write the variation as a common factor of that. This means we have to rewrite the 2nd and 4th term. Consider the 2nd term as an example. First we swap indices,

$$\eta^{\mu\nu}\partial_\sigma\partial_\rho h^{\rho\sigma}\delta h_{\mu\nu} = \eta^{\rho\sigma}\partial_\nu\partial_\mu h^{\mu\nu}\delta h_{\rho\sigma}.$$

Next we note that $\eta^{\rho\sigma}\delta h_{\rho\sigma} = \delta h = \eta_{\rho\sigma}\delta h^{\rho\sigma}$. This means the 2nd term becomes

$$\text{2nd term} = \eta_{\rho\sigma}\partial_\nu\partial_\mu h^{\mu\nu}\delta h^{\rho\sigma},$$

which is in terms of $\delta h^{\rho\sigma}$, as desired. The same can be done to the 4th term, and after bringing the $-\eta^{\mu\nu}$ back into the whole expression, the variation becomes

$$\delta S = \frac{1}{2}\int d^4x \left(-\partial_\rho\partial_\sigma h - \eta_{\rho\sigma}\partial_\mu\partial_\nu h^{\mu\nu} + \partial_\mu\partial_\rho h^\mu_\sigma + \partial_\mu\partial_\sigma h^\mu_\rho - \Box h_{\rho\sigma} + \eta_{\rho\sigma}\Box h\right)\delta h^{\rho\sigma}$$

The integrand must vanish, and we also note that, from equation (7.8), we have the definition of the linear Einstein tensor. Therefore,

$$G_{\rho\sigma} = 0.$$

This is the vacuum Einstein equation, to first order. To add in the matter part, it's actually identical to the derivation of the full Einstein equations (cf. chapter 4 in Carroll). Write the action as

$$S = \frac{1}{16\pi G}S_H + S_M$$

and we then get

$$G^{(0)}_{\mu\nu} = 8\pi G T_{\mu\nu}.$$

Problem 36. Gauge invariance in linearised gravity.

Let's do the actual calculations first before discussing what they really mean. We decompose the perturbation as so (7.16):

$$\begin{aligned} h_{00} &= -2\Phi \\ h_{0i} &= w_i \\ h_{ij} &= 2s_{ij} - 2\Psi\delta_{ij}, \end{aligned}$$

where Ψ encodes the trace of h_{ij}, and s_{ij} is traceless (7.17):

$$\Psi = -\frac{1}{6}\delta^{ij}h_{ij}$$

$$s_{ij} = \frac{1}{2}\left(h_{ij} - \frac{1}{3}\delta^{kl}h_{kl}\delta_{ij}\right).$$

Let's see how this decomposition changes under the gauge transformation given by $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu$. We first have

$$\begin{aligned} h_{00} &\to h_{00} + \partial_0\xi_0 + \partial_0\xi_0 \\ -2\Phi &\to -2\Phi + 2\partial_0\xi_0 \\ \Phi &\to \Phi + \partial_0\xi^0. \end{aligned}$$

where the sign change in the last line for ξ came from contracting with the metric component $\eta_{00} = -1$. Next we have

$$\begin{aligned} h_{0i} &\to h_{0i} + \partial_0 \xi_i + \partial_i \xi_0 \\ w_i &\to w_i + \partial_0 \xi_i + \partial_i \xi_0 \\ w_i &\to w_i + \partial_0 \xi^i - \partial_i \xi^0. \end{aligned}$$

We compute the spatial trace of $h^i_i = h$

$$\begin{aligned} \delta^{ij} h_{ij} &\to \delta^{ij} h_{ij} + \delta^{ij} \partial_i \xi_j + \delta^{ij} \partial_j \xi_i \\ \delta^{ij} h_{ij} &\to \delta^{ij} h_{ij} + 2\partial_i \xi_i \\ -6\Psi &\to -6\Psi + 2\partial_i \xi^i \\ \Psi &\to \Psi - \frac{1}{3} \partial_i \xi^i. \end{aligned}$$

Finally, the traceless part of h_{ij}:

$$\begin{aligned} \left(h_{ij} - \frac{1}{3}\delta^{kl} h_{kl} \delta_{ij}\right) &\to \left(h_{ij} - \frac{1}{3}\delta^{kl} h_{kl} \delta_{ij}\right) + \partial_i \xi_j - \frac{1}{3}\delta^{kl}\partial_k \xi_l \delta_{ij} + \partial_j \xi_i - \frac{1}{3}\delta^{kl}\partial_l \xi_k \delta_{ij} \\ 2s_{ij} &\to 2s_{ij} + 2\partial_{(i}\xi_{j)} - \frac{1}{3}\delta^{kl}\partial_k \xi_l \delta_{ij} - \frac{1}{3}\delta^{kl}\partial_l \xi_k \delta_{ij} \\ 2s_{ij} &\to 2s_{ij} + 2\partial_{(i}\xi_{j)} - \frac{1}{6}\partial_k \xi_k \delta_{ij} \\ s_{ij} &\to s_{ij} + \partial_{(i}\xi_{j)} - \frac{1}{3}\partial_k \xi^k \delta_{ij}. \end{aligned}$$

We have shown equation (7.33). Now, we can discuss gauge invariance. Carroll spells this out in more detail in section 7.1, but the essentials are this:

Consider two manifolds that are equipped with different metrics: (flat) background spacetime M_b and (curved) physical spacetime M_p. The metric $g_{\mu\nu}$ on the physical spacetime obeys Einstein's equations and we use a diffeomorphism ϕ to pull this metric back to flat spacetime (i.e. to move between the two spacetimes). We define the perturbation $h_{\mu\nu}$ to be the difference between the pullbacked and the flat metric. In this setting, gauge invariance amounts to a large number of possible diffeomorphisms ϕ, i.e. the decomposition $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ is not unique. The non-uniqueness grants us gauge freedom.

Analogously to electrodynamics, gauge freedom arises there since potentials are not unique. The EM fields are the related to *changes* in potentials, not the potential itself. This means if the potentials differ by some constant, you can define the difference in potentials to subtract that constant off. This is analogous to what's done in linear gravity: the diffeomorphisms, which cause the field $h_{\mu\nu}$, differ by some quantity, but in this case it's $2\epsilon\partial_{(\mu}\xi_{\nu)}$ (and not a scalar constant C).

Problem 37. Helicity.

Recall in general that the rotation matrix in the xy-plane is given by

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

This means our coordinate transformation will look like

$$C_{\mu\nu} = R^\alpha_\mu C_{\alpha\beta} R^\beta_\nu.$$

(Recall rotating a matrix in general).

Now, we know $C_{\alpha\beta}$ - it's equation (7.109). After cranking through all the algebra, you should find

$$C_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \cos(2\theta)h_+ + \sin(2\theta)h_\times & \cos(2\theta)h_\times - \sin(2\theta)h_+ & 0 \\ 0 & \cos(2\theta)h_\times - \sin(2\theta)h_+ & -\cos(2\theta)h_+ - \sin(2\theta)h_\times & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore the polarisation factors transform as

$$h_{R,L} \to h'_{R,L} = e^{\mp i2\theta} h_{R,L}$$

as per our redefinition $h_{R,L}$. This means GWs are of helicity ± 2.