FYS4160 - General Relativity
Problem Set 12 Solutions
Spring 2024

These solutions are credited to Jake Gordin, who wrote them in the years 2020-23.

If you spot any typos, mistakes, don't hesitate to contact me at halvor.melkild@fys.uio.no. For any physics related question please use the forum at astro-discourse.uio.no.

The idea of these solutions is to give you a sense of what a 'model' answer should be, and they also elaborate on some discussions from the help sessions. I try to make them "pedagogical": i.e. hopefully comprehensive and most steps should be explained.

Problem 38. Area in TT gauge.

As we have seen before, the area of a 2D surface is given by $A = \int \sqrt{-\tilde{g}} dxdy$, where $\tilde{g}$ is the induced metric on the surface. In the TT gauge, the line element is

$$ds^2 = -dt^2 + (1 + 2s_{ij})\, dx^i dx^j.$$

Therefore the induced metric will be of the form $\tilde{g}_{ij} = 1 + \mathcal{O}(h)$ (i.e. a function of s_{ij} and its derivatives.)

Thus, the determinant would be of the form $1 + \mathcal{O}\left(h^2\right)$ and we ignore the second order term because we are working in the linear regime. We see then that adding in a GW perturbation leaves the original area unchanged.

Problem 39. Energy of gravitational waves.

(a) Equation 7.9 is

$$\mathcal{L} = \frac{1}{2}\left[\left(\partial_\mu h^{\mu v}\right)\left(\partial_\nu h\right) - \left(\partial_\mu h^{\rho\sigma}\right)\left(\partial_\rho h^\mu_\sigma\right) + \frac{1}{2}\eta^{\mu\nu}\left(\partial_\mu h^{\rho\sigma}\right)\left(\partial_\nu h_{\rho\sigma}\right) - \frac{1}{2}\eta^{\mu\nu}\left(\partial_\mu h\right)\left(\partial_\nu h\right)\right].$$

The transverse-traceless (TT) gauge means, as the name implies, the perturbation is transverse, $\partial_\mu h^{\mu\nu} = 0$, and traceless, $h = 0$. This immediately reduces to action to[1]

$$\mathcal{L} = \frac{1}{2}\left[-\left(\partial_\mu h^{\rho\sigma}\right)\left(\partial_\rho h^\mu_\sigma\right) + \frac{1}{2}\eta^{\mu\nu}\left(\partial_\mu h^{\rho\sigma}\right)\left(\partial_\nu h_{\rho\sigma}\right)\right].$$

We note that since the wave is in the x_3-direction, $k_\rho = (\omega, 0, 0, \omega)$. We use the general solution for $h_{\mu\nu} = C_{\mu\nu} e^{ix_\rho k^\rho}$ to write

$$\partial_\rho h_{\mu\nu} = C_{\mu\nu}\partial_\rho e^{ik\cdot x} = ik_\rho C_{\mu\nu} e^{ik\cdot x} = ik_\rho h_{\mu\nu} \quad \text{thus, } \partial_1 h_{\mu\nu} = \partial_2 h_{\mu\nu} = \mathbf{0}$$

The first term can be written as $\eta^{\mu\nu}\left(\partial_\mu h^{\rho\sigma}\right)\left(\partial_\rho h_{\nu\sigma}\right)$. The index ρ has to be either 1 or 2 if the term is to be nonzero. But this means the derivative ∂_ρ is either $\partial_1 h_{\nu\sigma}$ or $\partial_2 h_{\nu\sigma}$, which we just showed is also zero. Hence the first term in the action is zero. We have left

$$\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\left(\partial_\mu h^{\rho\sigma}\right)\left(\partial_\nu h_{\rho\sigma}\right) = \frac{1}{2}\left(\partial_\mu h^{\rho\sigma}\right)\left(\partial^\mu h_{\rho\sigma}\right).$$

We also know the TT gauge fixes the nonzero parts of $h_{\mu\nu}$ to be $h_{11}, h_{12}, h_{21}, h_{22}$., and so if we sum over ρ, σ we find

$$\mathcal{L} \propto \left(\partial_\mu h^{\rho\sigma}\right)\left(\partial^\mu h_{\rho\sigma}\right) = \left(\partial_\mu h^{11}\right)\left(\partial^\mu h_{11}\right) + \left(\partial_\mu h^{12}\right)\left(\partial^\mu h_{12}\right) + \left(\partial_\mu h^{21}\right)\left(\partial^\mu h_{21}\right) + \left(\partial_\mu h^{22}\right)\left(\partial^\mu h_{22}\right)$$

[1] Note the matching indices for the transverse condition.

Since $h_{11} = -h_{22}$ and $h_{12} = h_{21}$, this becomes

$$\mathcal{L} \propto \left(\partial_\mu h^{11}\right)\left(\partial^\mu h_{11}\right) + \left(\partial_\mu h^{12}\right)\left(\partial^\mu h_{12}\right).$$

(b) We have

$$\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\left(\partial_\mu h^{\rho\sigma}\right)\left(\partial_\nu h_{\rho\sigma}\right).$$

We want to compute

$$t_\mu^\nu = h_{\rho\sigma,\mu}\frac{\partial \mathcal{L}}{\partial h_{\rho\sigma,\nu}} - \delta_\mu^\nu \mathcal{L}$$

This is straightforward since

$$\frac{\partial \mathcal{L}}{\partial h_{\rho\sigma,\nu}} = \frac{1}{2}\eta^{\mu\nu}\left(\partial_\mu h^{\rho\sigma}\right)$$

Therefore

$$t_\mu^\nu = \frac{1}{2}\partial_\mu h_{\rho\sigma}\partial^\nu h^{\rho\sigma} - \delta_\mu^\nu \mathcal{L}.$$

The energy flux in the ith direction (recall the breakdown of $T_{\mu\nu}$ in problem set 7) is t_{0i}. Therefore, noting that $\delta_\mu^\nu = 0$ if $\mu \neq \nu$, we have

$$t_{0i} = \frac{1}{2}\partial_0 h_{\rho\sigma}\partial_i h^{\rho\sigma}.$$

Using the results from part (a), in terms of the wave's being in the x_3-direction, this becomes

$$t_{03} = \omega^2(h_{11}^2 + h_{12}^2) = \omega^2(h_+^2 + h_\times^2).$$

(c) Eq. (7.182) is

$$h_{ij} = \frac{2G}{r}\frac{d^2 J_{ij}}{dt^2}$$

Here J_{ij} is the reduced quadrupole moment. I've omitted the notation telling us we're in the TT gauge. The result follows immediately from part (b):

$$t_{0i} = \frac{1}{2}\partial_0 h_{\rho\sigma}\partial_i h^{\rho\sigma}.$$

We compute first

$$\partial_0 h_{ij} = \frac{2G}{r}\frac{d^3 J_{ij}}{dt^3} \quad \text{and} \quad \partial_k h_{ij} = -\partial_0 h_{ij} = -\frac{2G}{r}\frac{d^3 J_{ij}}{dt^3}.$$

where $\partial_k h_{ij} = -\partial_0 h_{ij}$ follows because $x_\rho k^\rho = -t\omega + x_3\omega$ for a x_3-directional wave and because we are far away from the source (cf. Carroll equation (7.183)). Thus we get for the energy flux,

$$t_{0i} = -\frac{2G^2}{r^2}\frac{d^3 J_{ij}}{dt^3}\frac{d^3 J^{ij}}{dt^3}.$$

Problem 40. The LIGO observation.

The energy density is contained in the t_{00} term. However, we can also say

$$t_{00} = t_{33} = -t_{03}$$

because of our TT-gauge (check this explicitly if you remain unsure). We also note that in the original Einstein-Hilbert action, there is a factor of $1/16\pi G$ that we left out since we only cared about proportionality in part (a). Restoring it, and noting the factor of a 1/2 we picked up, we get

$$t_{03} = \frac{1}{32\pi G}\omega^2(h_+^2 + h_\times^2).$$

The way the two polarisation terms, $h_+^2, h_\times^2$ distort a ring of particles is collectively called the *strain*. This means we have

$$t_{03} = \frac{1}{32\pi G}\omega^2 f^2.$$

Restoring SI units, we have

$$t_{03} = \frac{c^2}{32\pi G}\omega^2 f^2.$$

Now consider hypothetically a graviton. It can't be constrained to a region smaller than its wavelength; this gives the rough volume as $\lambda^3 = c^3/\nu^3$, where we've used the classic (and classical) relation $c = \lambda\nu$. The energy density is therefore

$$\rho = \frac{h\nu^4}{c^3}$$

where here h is Planck's constant. The frequency here can be rewritten using $\omega = 2\pi\nu \implies \nu^4 = \omega^4/(2\pi)^4$ to give

$$\rho = \frac{h\omega^4}{(2\pi)^4 c^3}.$$

The ratio of the GW energy density to a single graviton is therefore

$$\mathcal{E} \equiv \frac{t_{03}}{\rho} = \frac{c^5 f^2 \pi^3}{2Gh\omega^2}.$$

We choose parameters $f \sim 10^{-21}$ and $\omega \sim 750\text{Hz}$ and we find $\mathcal{E} \sim 10^{40}$. We thus require roughly a sensitivity improvement on the order of 10^{40} if we are to hope to detect individual gravitons (at least with LIGO) (A bit of a tall order).

Note that it's not entirely clear that graviton energy can be given as $h\nu$, as it is for photons. It's not known currently what factors contribute to the "bare" graviton energy; but since we don't know, we're assuming for now that we can use the same relation we use for photons (if only this logic worked in all parts of life).