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http://www.uio.no/studier/emner/matnat/fys/FYS4170/h16/

Mid-term exam

Lecture autumn 2016: Relativistic quantum field (FYS4170)

⇝ *Answers must be handed in latest* **Monday, 31 October 2016, 10:00am***; you can use the corresponding box (labelled with the course name and code) at the administrative office of the Physics Department. Please write your candidate number (not name!) on the top of the front page.*

Maximal number of available points: 55.

Problem 1 (*3 points*)
Use the Dirac algebra to simplify the following expressions using Dirac matrices:

a) $\gamma^\mu \gamma_\mu$

b) $\gamma^\mu \gamma^\nu \gamma_\mu$

c) $\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu$

Problem 2
The field operator ψ for the quantized Dirac field can be written as

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_\mathbf{p}}} \sum_s \left(a^s_\mathbf{p} u^s(p) e^{-ip\cdot x} + b^{s\,\dagger}_\mathbf{p} v^s(p) e^{ip\cdot x} \right) .$$

a) What are the meaning and the main properties of the various components of this expression? (*5 points*)

b) Recap, in your own words, how we arrived at this expression from first principles, and state the most important intermediate results! [don't use more than 0.5 – 1 page on this!]
(*5 points*)

Problem 3 (*6 points*)
What are the properties of the QED Lagrangian under the parity transformations C, P, and T? Use the explicit representations of these operations in terms of gamma matrices to derive your result (as well as the knowledge that the electromagnetic potential A^μ is a 4-*vector*).

Problem 4 (*5 points*)

Recall that the vector current $j^\mu_V \equiv \bar{\psi}\gamma^\mu\psi$ is divergence-free (i.e. $\partial_\mu j^\mu_V = 0$) for any field $\psi(x)$ that satisfies the Dirac equation. Calculate the conserved charge in terms of creation and annihilation operators! What is the value for a single fermion (and antifermion) of a given momentum $\mathbf{p}$?

Problem 5
Consider the following theory of two real scalar fields ϕ and Φ, where the former is massless:

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi\right)^2 + \frac{1}{2}\left(\partial_\mu\Phi\right)^2 - \frac{1}{2}m_\Phi^2\Phi^2 - \frac{\mu}{2}\Phi\phi^2 \,.$$

a) What are the Feynman rules for the interactions in this theory? What is the dimension of the constant μ? (*2 points*)

b) Calculate, to leading order in perturbation theory, the decay rate of the field Φ! (*3 points*)

c) Draw all Feynman diagrams that appear in principle when considering the decay of Φ at next-to-leading order (by applying Wick's theorem). Which of those diagrams contribute to the invariant matrix element $\mathcal{M}$? Among those diagrams, are there vacuum diagram contributions – and why do those not contribute to $\mathcal{M}$? Which diagrams contain contributions to the self-energies of the scalar fields, and why do those not contribute to $\mathcal{M}$? (*6 points*)
[Hint: You may have to consider more than one possible final state.]

Problem 6 (*8 points*)
In the lecture, we have in some detail studied the QED process of Compton scattering. Consider instead the same process in Yukawa theory, i.e. replace the interaction Lagrangian $\mathcal{L}_{\text{int}} = -e\overline{\psi}\gamma^\mu\psi A_\mu$ with $\mathcal{L}_{\text{int}} = -g_e\overline{\psi}\psi\phi$ and consider the scattering of electrons with a very light ($m_\phi \ll m_e$) scalar particle. Compute the unpolarized, differential cross section $d\sigma/d\Omega$ and discuss the differences to the Klein-Nishima and Thompson formulae!

Problem 7

In order to describe the scattering of an electron, or positron, in a time-independent classical electromagnetic field one can simply replace the QED vertex rule $-ie\gamma^\mu \to -ie\gamma^\mu \tilde{A}_\mu(\mathbf{q})$. Here, $A_\mu(\mathbf{x})$ is the classical electromagnetic potential, $\tilde{A}_\mu(\mathbf{q})$ its Fourier transform and $q \equiv p_f - p_i$ the difference between incoming and outgoing *fermion* momenta. For a potential that is not only time-independent but also localized in space, the scattering cross section can then be written as

$$d\sigma = \frac{1}{2|\mathbf{p}_i|}\frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\left|\mathcal{M}(p_i \to p_f)\right|^2 (2\pi)\delta(E_f - E_i)\,.$$

[See P&S, problem 4.4, for a motivation of all these statements. Note that the amplitude is not dimensionless in this case, like for $2 \to 2$ *scattering!]*

a) Show that the last expression is equivalent to (*2 points*)

$$\frac{d\sigma}{d\Omega} = \frac{1}{16\pi^2}\left|\mathcal{M}(p_i \to p_f)\right|^2$$

b) Compute the scattering amplitude $\mathcal{M}$ for the scattering of an electron in the Coulomb potential created by a nucleus of charge Z, i.e. $A^\mu = (Ze/4\pi r, \mathbf{0})$. How does this expression look like for the scattering of a positron? (*4 points*)
(Hint: The Fourier transform of the Coulomb potential is most easily calculated by adding a regulating factor $e^{-\mu r}$ *to the potential, and then sending the 'photon mass'* μ *to zero at the end of the calculation.)*

c) Using the above expressions, calculate the spin-averaged cross-section for the scattering of an electron in a Coulomb potential. The result is known as the *Mott formula*. Take the non-relativistic limit of this expression to obtain a well-known expression for the scattering of charged particles obtained earlier by *Rutherford*. (*6 points*)