Torsten Bringmann (torsten.bringmann@fys.uio.no)
http://www.uio.no/studier/emner/matnat/fys/FYS4170/h18/

FYS4170: Preliminaries

There are many key mathematical and physical concepts that quantum field theory rests on. These notes summarize some of the *things that you absolutely need to know* before being able to successfully follow this course. It also fixes our notation and conventions (we will use the same as in the book by Peskin & Schroeder). If you feel even slightly uncomfortable with any of the below concepts or terms, you should as soon as possible review a suitable book or your notes of previous courses.

We will throughout use 'natural units' where $\hbar = c = 1$ (though these constants are kept at some places for pedagogic reasons). All dimensionfull quantities can then be expressed in terms of energy, the typical choice of unit being GeV. For example,

$$\begin{aligned} 1\,\text{GeV} &= 1.783 \times 10^{-24}\,\text{g}\,[\times c^2] \\ 1\,\text{GeV}^{-1} &= 1.973 \times 10^{-14}\,\text{cm}\,[\times 1/\hbar c] = 6.581 \times 10^{-25}\,\text{s}\,[\times 1/\hbar] \\ 1\,\text{GeV}^{-2} &= 3.894 \times 10^{-28}\,\text{cm}\,[\times 1/\hbar^2 c^2] \equiv 3.894 \times 10^{-4}\,\text{barn}\,[\times 1/\hbar^2 c^2] \end{aligned}$$

1 Special relativity

The time t and the position vector $\mathbf{x}$ (on the blackboard we will use $\vec{x}$) can be combined into a four-vector in Minkowski space $x^\mu = (t, \mathbf{x}) = (x^0, x^1, x^2, x^3)$. We use small greek letters to denote space-time indices and small roman letters to denote spatial indices of such four-vectors, i.e. $\mu, \nu, ... = 0, 1, 2, 3$ and $i, j, ... = 1, 2, 3$.[1] Throughout the course, the **summation convention** is used, in which repeated indices are always summed over. This implies that any index that appears twice in an expression is a dummy index (i.e. its name has no importance: $x^\mu x_\mu = x^\rho x_\rho = x^0 x_0 + x^1 x_1 + x^2 x_2 + x^3 x_3$) and that no index can appear more than twice. For spacetime-indices, it is furthermore very important to keep track of whether they are up or down; in particular, you should never encounter expressions like $a^\mu b^\mu$ (see also below the comments about scalar products).

In Minkowski space, the **metric** tensor g (aka $g_{\mu\nu}$, see the footnote) is given by

$$g_{\mu\nu} = \eta_{\mu\nu} \equiv \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} = \left(g^{-1}\right)_{\mu\nu} \equiv g^{\mu\nu}. \tag{1}$$

We use the metric $g_{\mu\nu}$ to lower indices, and the inverse metric $g^{\mu\nu}$ to raise indices. For example, $x_\mu \equiv g_{\mu\nu} x^\nu = (t, -\mathbf{x})$ and $x^\mu = g^{\mu\nu} x_\nu$. Note that while the displacement vector $x^\mu = (t, \mathbf{x})$ is 'naturally raised', the derivative vector is naturally lowered:

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial x^0}, \nabla\right), \quad \text{so} \quad \partial^\mu = \left(\frac{\partial}{\partial x^0}, -\nabla\right). \tag{2}$$

[1] For four-vectors we sometimes follow the standard, if somewhat confusing, convention of writing A^μ or A_μ instead of 'A'; we thus use the *same* notation for a vector and its components, and it is only the *context* that determines the correct interpretation. (This convention allows a simple distinction between covariant ('lower indices') and contravariant ('upper indices') vectors). A Euclidian three-vector, in contrast, is always denoted as $\mathbf{A}$ (or $\vec{A}$), while its components are denoted with A^i.

A **four-vector** in general is any object that transforms under a Lorentz transformation Λ, described by a 4×4 matrix $\Lambda^\mu{}_\nu$, according to

$$A^\mu \to A'^\mu = \Lambda^\mu{}_\nu A^\nu \qquad (\text{and thus } A_\mu \to A'_\mu = \Lambda_\mu{}^\nu A_\nu)\,. \tag{3}$$

A **Lorentz transformation** is by definition any transformation that leaves the 'length' of a vector in Minkowski space invariant:[2]

$$\eta_{\mu\nu} x^\mu x^\nu \stackrel{!}{=} \eta_{\mu\nu} x'^\mu x'^\nu = \eta_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma x^\rho x^\sigma\,. \tag{4}$$

(More generally, a Lorentz **tensor** is an object with several indices, $T^{\mu_1\mu_2\cdots}_{\nu_1\nu 2\ldots}$, that all transform as above.) This implies that *any* scalar product involving four-vectors is invariant (i.e. a 'scalar') under Lorentz transformations:

$$A \cdot B \equiv A^\mu B_\mu = A_\mu B^\mu = A^0 B^0 - \mathbf{A}\cdot\mathbf{B} = const. \tag{5}$$

Note that a scalar product always involves a contravariant and a covariant vector. Lorentz transformations contain standard 3D rotations as well as Lorentz boosts.

Example: For a boost along the x-axis, Λ takes the form

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{6}$$

with $\beta \equiv v/c$ denoting the relative velocity of the two frames and $\gamma \equiv (1-\beta^2)^{-1/2}$. Using Eq. (3), this implies the familiar transformation $x \to x'$ with

$$t \;\to\; t' = \gamma(t - \beta x) \tag{7}$$
$$x \;\to\; x' = \gamma(x - \beta t) \tag{8}$$
$$y \;\to\; y' = y \tag{9}$$
$$z \;\to\; z' = z \tag{10}$$

The most important four-vector that we will encounter, besides the position vector, is the **four-momentum**. It satisfies $p^2 \equiv p \cdot p = m^2$, where m is the invariant mass of the particle. Writing the four-momentum in terms of the energy E and the 3-momentum $\mathbf{p}$,

$$p^\mu = (E, \mathbf{p})\,, \tag{11}$$

we thus directly get the famous relativistic dispersion relation for a massive particle:

$$E^2 - \mathbf{p}^2 = m^2\,. \tag{12}$$

[2] Recall that equations like this actually describe matrix multiplications. Without index notation, it would read

$$x^T \cdot \eta \cdot x \stackrel{!}{=} x'^T \cdot \eta \cdot x' = \Lambda^T \cdot x^T \cdot \eta \cdot \Lambda \cdot x\,.$$

Maxwell's equations, using Heaviside-Lorentz conventions[3], read

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \rho \,, & (13)\\ \nabla \times \mathbf{E} &= -\partial_t \mathbf{B} \,, & (14)\\ \nabla \cdot \mathbf{B} &= 0 \,, & (15)\\ \nabla \times \mathbf{B} &= \mathbf{J} + \partial_t \mathbf{E} \,, & (16) \end{aligned}$$

where ρ is the charge and $\mathbf{J}$ the current density. By realizing that the electromagnetic potentials Φ and $\mathbf{A}$ (from which the electric and magnetic fields can be constructed as $\mathbf{E} = -\nabla\phi - \partial_t \mathbf{A}$ and $\mathbf{B} = \nabla \times \mathbf{A}$) can be combined into a *four vector*

$$A^\mu \equiv (\phi, \mathbf{A}) \,, \tag{17}$$

the above four Maxwell equations can be brought into the following elegant form:

$$\begin{aligned} \partial_\mu F^{\mu\nu} &= e j^\mu \,, & (18)\\ \epsilon^{\mu\nu\rho\sigma} \partial_\nu F_{\rho\sigma} &= 0 \,. & (19) \end{aligned}$$

Here, we have introduced the *field tensor* $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ and the *current density* four-vector $j^\mu \equiv (\rho, \mathbf{j})/e$. The advantage of the tensor notation is not only its compactness but also that it now becomes manifest that Maxwells equations are Lorentz invariant (i.e. they take the same form in any inertial system).

In vacuum ($j^\mu = 0$), and adopting the Lorenz gauge ($\partial_\mu A^\mu = 0$), Maxwells equations reduce to a set of four wave equations (can you show this?),

$$\Box A^\mu \equiv \partial_\nu \partial^\nu A^\mu = 0 \,, \tag{20}$$

for the components of the (four-)vector potential. The solution to this equation is

$$A^\mu = \epsilon^\mu \, e^{ik\cdot x} \,, \tag{21}$$

where $k = (\omega, \mathbf{k})$, with $|\mathbf{k}| = \omega$, is the wave-vector (and hence its four-momentum). The constant ϵ^μ is known as the *polarization* (or polarization vector) of the wave. It is subject to two constraints,[4]

$$\begin{aligned} k \cdot \epsilon &= k_\mu \epsilon^\mu = 0, & (22)\\ \mathbf{k} \cdot \epsilon &= k_i \epsilon^i = 0 \,, & (23) \end{aligned}$$

leaving thus *two* linearly independent polarization patterns (e.g. left- and right circular polarization).

[3] In these conventions, which we will use throughout, the Coulomb potential of a point charge is $Q/4\pi r$ and the fine-structure constant $\alpha = e^2/4\pi \approx 1/137$.

[4] The first follows from the Lorentz gauge condition. The second constraint is equivalent to saying that electromagnetic waves are *transverse* and can directly be derived from the first Maxwell equation (try to do this!).

2 Classical dynamics: Lagrange & Hamilton

The state of a classical physical system at a given time t can be fully characterized by a set of *generalized coordinates* q_a, with $a = 1, ..., N$ (for an unconstrained system, N equals three times the number of particles), and their associated *velocities* $\dot{q}_a \equiv dq_a/dt$. The *principle of least action* requires that the true path $q(t)$ taken by the system (the physical trajectory) is the one that extremizes the *action*, i.e. the functional

$$S[q] \equiv \int_{t_1}^{t_2} L(q_a, \dot{q}_a)\, dt\,. \tag{24}$$

The whole dynamics is thus governed by the *Lagrangian* $L = L(q_a, \dot{q}_a)$.[5] Demanding $\delta S = 0$, in particular, leads to the **Euler-Lagrange equations** of motion:

$$\frac{dL}{dq_a} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_a} = 0\,. \tag{25}$$

Example. For a single particle in a potential $V(\mathbf{x})$, we have $L = \frac{1}{2}m\dot{\mathbf{x}}^2 - V$ in Cartesian coordinates. Eq. (25) then reproduces the familiar $m\ddot{\mathbf{x}} = -\nabla V$.

For each generalized coordinate q_a, the **canonically conjugated momentum** p_a is defined as

$$p_a \equiv \frac{\partial L(q, \dot{q})}{\partial q_a}\,. \tag{26}$$

A Legendre transformation of the Lagrangian then leads to the *Hamiltonian* H, which no longer depends on the velocities:[6]

$$H(p, q) \equiv p^a \dot{q}_a(p, q) - L[q, \dot{q}(p, q)]\,. \tag{27}$$

In this formulation, the system is described by the **Hamilton equations** of motion,

$$\dot{q}_i = \frac{\partial H}{\partial p_i}\,, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}\,, \tag{28}$$

i.e. $2N$ coupled first-order ODEs instead of the N second-order ODEs in Eq. (25).

Example. Considering again a single particle in a potential, we have $H = \mathbf{p}\cdot\dot{\mathbf{x}} - L = \frac{1}{2m}\mathbf{p}^2 + V$. The Hamilton equations then give as expected $\dot{\mathbf{q}} = \mathbf{p}/m$ and $\dot{\mathbf{p}} = -\nabla V$.

An important quantity in classical mechanics is furthermore the **Poisson bracket**, which is defined as

$$\{f, g\} \equiv \frac{\partial f}{\partial q^a}\frac{\partial g}{\partial p_a} - \frac{\partial f}{\partial p_a}\frac{\partial g}{\partial q^a} \tag{29}$$

for any two functions $f(q, p)$ and $g(q, p)$ on phase space. The time evolution of any physical quantity $f = f(t, q, p)$ then takes the simple form

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial H}{\partial p^a}\frac{\partial f}{\partial q_a} - \frac{\partial H}{\partial q_a}\frac{\partial f}{\partial p^a} = \partial_t f - \{H, f\}\,, \tag{30}$$

[5] For *open* systems, i.e. if an external force is present, L can also have an explicit time-dependence.

[6] Note that this is only possible if Eq. (26) can be solved for $\dot{q}_a$. Normally, this requires $\det \partial L/\partial\dot{q}_a\partial q_b \neq 0$.

where the explicit time-dependence of f – and hence the ∂_t term – never appears in applications relevant to our context. Another important aspect, which significantly helps to construct the quantum theory, are the properties of the phase-space variables

$$\{q_a, p_b\} = \delta_{ab} \qquad \text{and} \qquad \{q_a, q_b\} = \{p_a, p_b\} = 0\,. \tag{31}$$

3 Quantum mechanics

In classical mechanics, the state of a physical system is fully specified by its phase-space variables q and p. In quantum mechanics, on the other hand, the state of a system is described in a conceptually very different way, namely by a vector $|\psi\rangle$ in Hilbert space. The **canonical quantization** of a classical system consists in promoting phase-space functions $f(q,p)$ to operators $\hat{f}$ that act on those quantum states. These operators are constructed such that they obey commutation relations instead of the classical Poisson brackets:

$$\{\,,\,\}_{\text{classical}} \to -\frac{i}{\hbar}\,[\,,\,]_{\text{quantum}}\,. \tag{32}$$

Note that this prescription is not unique due to ordering ambiguities, i.e. there are in general several operators $\hat{f}$ that correspond to a given classical function f. In order to facilitate notation, we will typically leave out the hat when it is clear from the context that a given quantity is an operator.

Example. From the above prescription, we obtain immediately the familiar relations for the position and momentum operators,

$$[\hat{q}_a, \hat{p}_b] = i\hbar\,\delta_{ab} \qquad \text{and} \qquad [\hat{q}_a, \hat{q}_b] = [\hat{p}_a, \hat{p}_b] = 0\,. \tag{33}$$

In position space, e.g., the **momentum operator** is explicitly given by $\hat{\mathbf{p}} = -i\hbar\nabla$. With $\hat{E} = i\hbar\partial_t$, this can be combined into (note sign and index position!)

$$\hat{p}^\mu = i\partial^\mu\,. \tag{34}$$

The plane wave $e^{-ik\cdot x}$, i.e. the wavefunction describing a single free particle, thus has momentum $+k^\mu$ since $\hat{p}^\mu\left(e^{-ik\cdot x}\right) = k^\mu e^{-ik\cdot x}$.

The advantage of the abstract Dirac (or **bra-ket**) **notation** is that it is independent of the *representation*, i.e. independent of the choice of basis. Once we choose a complete orthonormal Hilbert-space basis $|n\rangle$, satisfying $\langle n|m\rangle = \delta_{nm}$ and $\sum |n\rangle\langle n| = \mathbf{1}$, we can always expand the state as

$$|\psi\rangle = \sum |n\rangle \underbrace{\langle n|\psi\rangle}_{\equiv c_n} = \sum c_n |n\rangle \qquad \text{and thus } \langle\psi| = \sum c_n^*\langle n|\,. \tag{35}$$

For a continuous rather than discrete basis, this works very similar. A typical example is the position space representation, where $|\mathbf{x}\rangle$ denote eigenvectors of the position operator: $\hat{\mathbf{q}}|\mathbf{x}\rangle = \mathbf{x}|\mathbf{x}\rangle$. The orthonomality and completeness relations then become

$$\langle\mathbf{x}|\mathbf{y}\rangle = \delta^{(3)}(\mathbf{x}-\mathbf{y}) \qquad \text{and} \qquad \int d^3x\,|\mathbf{x}\rangle\langle\mathbf{x}| = \mathbf{1}\,. \tag{36}$$

This can be used to calculate the scalar product of any two vectors $|\psi_1\rangle$ and $|\psi_2\rangle$ as

$$\langle\psi_1|\psi_2\rangle = \int d^3x\, \langle\psi_1|\mathbf{x}\rangle\langle\mathbf{x}|\psi_2\rangle = \int d^3x\, \psi_1(\mathbf{x})^*\psi_2(\mathbf{x}) = \langle\psi_2|\psi_1\rangle^*\,, \tag{37}$$

where $\psi(t,\mathbf{x}) = \langle\mathbf{x}|\psi\rangle$ is the familiar Schrödinger *wavefunction* in position space (assuming that $|\psi\rangle = \int d^3x\, \psi(\mathbf{x})|\mathbf{x}\rangle$ is given in the Schrödinger picture, see below). In momentum space, $|\psi\rangle = \int d^3p\, (2\pi)^{-3}\, \psi(\mathbf{p})|\mathbf{p}\rangle$ leads to the identical scalar product. For every observable O, there is further an operator $\hat{O}$ such that the **expectation value** of O for a state $|\psi\rangle$ is $\langle O\rangle = \langle\psi|\hat{O}|\psi\rangle = \langle\psi|\hat{O}\psi\rangle = \langle\hat{O}^\dagger\psi|\psi\rangle$ $(= \int d^3x\, \psi^*(\mathbf{x})\hat{O}\psi(\mathbf{x})$ when evaluated in the real space basis).

In the classical case, it is the Hamiltonian H that governs the time evolution of a physical system, c.f. Eq. (30). The dynamics in quantum mechanics, in analogy, is governed by the *Hamiltonian operator* $\hat{H}$ and can be described in different ways. In the **Schrödinger picture**, which you likely are most familiar with, all operators $\hat{O}$ are time-independent and the states $|\psi\rangle$ evolve according to the *Schrödinger equation*

$$i\hbar\frac{d|\psi\rangle_S}{dt} = \hat{H}|\psi\rangle_S\,. \tag{38}$$

An equivalent description is provided by the **Heisenberg picture**, in which the states are time-independent while the operators evolve in time according to

$$\frac{d\hat{O}_H}{dt} = \frac{i}{\hbar}\left[\hat{H}, \hat{O}_H\right]\,, \tag{39}$$

just as expected from Eqns. (30,32). Changing between these two pictures is simple:

$$|\psi\rangle_H = e^{i\hat{H}t/\hbar}|\psi\rangle_S\,, \tag{40}$$

$$\hat{O}_H = e^{i\hat{H}t/\hbar}\,\hat{O}_S\,e^{-i\hat{H}t/\hbar}\,. \tag{41}$$

This implies, e.g., that $\hat{H}_S = \hat{H}_H = \hat{H}$ and that the expectation value of any operator $\hat{O}$ is the same in both pictures: ${}_S\langle\psi|\hat{O}_S|\psi\rangle_S = {}_H\langle\psi|\hat{O}_H|\psi\rangle_H$. In the course, we will use both pictures extensively, as well as a hybrid version known as *interaction picture*.

The quantum treatment of the **harmonic oscillator**, finally, is of central importance to quantum field theory, too. The eigenstates of the Hamiltonian $\hat{H} = \hat{\mathbf{p}}^2/(2m) + \frac{1}{2}m\omega^2\hat{\mathbf{q}}^2$ are most conveniently found by introducing **ladder operators**

$$a_j \equiv \sqrt{\frac{m\omega}{2}}\left(\hat{q}_j + i\frac{\hat{p}_j}{m\omega}\right) \qquad \text{and thus} \qquad a_j^\dagger \equiv \sqrt{\frac{m\omega}{2}}\left(\hat{q}_j - i\frac{\hat{p}_j}{m\omega}\right)\,, \tag{42}$$

which because of Eq. (33) satisfy $[a_i, a_j^\dagger] = 1$ and $[a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0$. This allows to rewrite $\hat{H} = \omega\sum_i\left(a_i^\dagger a_i + \frac{1}{2}\right)$, where the number operator $N_i \equiv a_i^\dagger a_i$ has integer eigenvalues n_i, i.e. $a_i^\dagger a_i|n_i\rangle = n_i|n_i\rangle$. The *vacuum* $|0\rangle$ (i.e. the lowest energy state) is given by $a_i|0\rangle = 0$, and all other states are 'created' from this because $a_i^\dagger|n_i\rangle \propto |n_i{+}1\rangle$.

4 Mathematical tools

4.1 Fourier transforms

We will frequently change between position and momentum space, with the following convention for the Fourier transforms relating those two:

$$f(x) = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot x} \tilde{f}(k)\,, \tag{43}$$

$$\tilde{f}(k) = \int d^4x\, e^{ik\cdot x} f(x)\,. \tag{44}$$

Factors of 2π will thus always appear in the momentum integrals; for 3D integrals we need to replace $k \cdot x \to -\mathbf{k}\cdot\mathbf{x}$ (note the minus sign!). Fourier transforms can help to find Green's functions (see the example below).

4.2 Dirac delta function

The Dirac delta 'function' $\delta(x)$ is not actually a function but a distribution, and is defined by $\delta(x) = 0$ for all $x \neq 0$, and $\int dx\, \delta(x) = 1$. An alternative definition is given in terms of the Heaviside step function $\Theta(x)$:

$$\delta(x) \equiv \frac{d}{dx}\Theta(x)\,, \qquad \text{where} \quad \Theta(x) \equiv \begin{cases} 0 & \text{for} \quad x < 0 \\ 1 & \text{for} \quad x > 0 \end{cases}\,. \tag{45}$$

Useful relations that appear often include

$$\int dx\, f(x)\, \delta\,(x-y) = f(y)\,, \tag{46}$$

$$\int dx\, f(x)\, \delta'\,(x) = -f'(0)\,, \tag{47}$$

$$\int dx\, e^{ikx} = 2\pi\, \delta(k)\,, \tag{48}$$

$$\delta(f(x)) = \sum_i \frac{\delta(x-x_i)}{|f'(x_i)|}\,, \tag{49}$$

where the second equation can be understood from integration by parts, and the last one only holds if the function $f(x)$ only has 1st order zeros (at $x = x_i$). Analogous definitions and properties of the delta function apply in higher dimensions.

4.3 Green's functions

Green's 'functions' (which often also rather are distributions), are an important tool to solve differential equations of the form

$$L_x f(x) = g(x)\,. \tag{50}$$

Here, L_x is a linear, differential operator in x, f the function to be determined and g an arbitrary function that constitutes the inhomogeneous part of the ODE. A Green's function G of L_x is by definition any – often not uniquely determined – solution of

$$L_x G(x,y) = -\delta(x-y)\,. \tag{51}$$

A Green's function is thus, in a loose sense, the *inverse* of L_x. If L_x is translation invariant (i.e. it does not change for $x \to x + x_0$, as is the case in most physics' applications), the Green's function can in fact always be written as

$$G(x,y) = G(x-y)\,. \tag{52}$$

If G is known, then a solution of Eq. (50) for (almost) *arbitrary* $g(x)$ is given by

$$f(x) = -\int dy\, G(x-y) g(y)\,, \tag{53}$$

which directly follows from the definitions of the Green's and Dirac delta functions. NB: With $f(x)$ being a solution to Eq. (50), $f(x) + \sum c_i f_{i,\mathrm{hom}}(x)$ is also a solution (where $f_{i,\mathrm{hom}}$ represent the homogenous solutions, i.e. those for which $L_x f_{i,\mathrm{hom}} = 0$)!

Example. Consider Eq. (13) for a static field, which is known as Poisson's equation:

$$\nabla^2 \phi(\mathbf{x}) = -\rho(\mathbf{x})\,. \tag{54}$$

Thus, $L_x = \nabla^2 = \partial^i \partial_i$ and $g = -\rho$ in the above notation. Writing G and δ in terms of their Fourier-transforms, Eq. (51) therefore reads

$$\nabla^2 \int \frac{d^3k}{(2\pi)^3}\, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\, \tilde{G}(\mathbf{k}) = -\int \frac{d^3k}{(2\pi)^3}\, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\, |\mathbf{k}|^2\, \tilde{G}(\mathbf{k}) = -\int \frac{d^3k}{(2\pi)^3}\, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\,. \tag{55}$$

Hence, $\tilde{G}(\mathbf{k}) = 1/|\mathbf{k}|^2$ and therefore

$$\begin{aligned} G(\mathbf{x}-\mathbf{y}) &= \int \frac{d^3k}{(2\pi)^3}\, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\, \tilde{G}(\mathbf{k}) && (56)\\ &= \frac{1}{4\pi^2}\int_{-1}^{1} d\cos\theta \int_0^\infty d|\mathbf{k}|\, e^{i\cos\theta|\mathbf{k}||\mathbf{x}-\mathbf{y}|} && (57)\\ &= \frac{1}{8\pi^2}\int_{-1}^{1} d\cos\theta \underbrace{\int_{-\infty}^{\infty} d|\mathbf{k}|\, e^{i\cos\theta|\mathbf{k}||\mathbf{x}-\mathbf{y}|}}_{2\pi\,\delta(\cos\theta|\mathbf{x}-\mathbf{y}|)=2\pi\,\delta(\cos\theta)/|\mathbf{x}-\mathbf{y}|} && (58) \end{aligned}$$

Inserting the resulting $G(\mathbf{x}-\mathbf{y}) = 1/4\pi|\mathbf{x}-\mathbf{y}|$ in Eq. (53), we find the familiar solution of Eq. (54): The total potential is obtained by summing over all sources,

$$\phi(\mathbf{x}) = \int d^3y\, \frac{\rho(\mathbf{y})}{4\pi|\mathbf{x}-\mathbf{y}|}\,. \tag{59}$$

4.4 Complex Analysis

Complex, differentiable (i.e. *holomorphic* or *analytic*) functions exhibit a number of fascinating properties. One of the central results of complex analysis is the **residue theorem**, which we will make use of at some central places. Let Γ be a positively oriented, simple closed contour in the complex plane (i.e. anti-clockwise, winding number 1). For any function $f(z)$ that is analytic inside (and on) Γ except for a finite number of singular points $z_1, ..., z_n$, we then have

$$\oint_\Gamma f(z)\, dz = 2\pi i \sum_{i=1}^{n} \mathrm{Res}(f, z_i)\,. \tag{60}$$

Here, $\mathrm{Res}(f, z_i)$ denotes the *residue* of f at z_i, which can be defined as the coefficient a_{-1} of the *Laurent expansion* around this point

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_i)^n\,. \tag{61}$$

In practice, the following rules help to determine the residue:

- If f has a simple (i.e. 1st order) *pole* at z_i: $\mathrm{Res}(f, z_i) = \lim\limits_{z\to z_i}\ (z - z_i) f(z)$
- If f has a 1st order *zero* at z_i: $\mathrm{Res}(1/f, z_i) = 1/f'(z_i)$

- If f has a pole of nth order at z_i: $\operatorname{Res}(f, z_i) = \frac{1}{(n-1)!} \lim_{z \to z_i} \frac{\partial^{n-1}}{\partial z^{n-1}} \left[(z - z_i)^n f(z)\right]$

Note that the residue theorem includes Cauchy's famous *integral theorem*, which states that $\oint f(z)\, dz = 0$ for every holomorphic function.

Acknowledgements. This write-up draws significantly from the introductory part of the excellent lecture notes provided by David Tong at `http://www.damtp.cam.ac.uk/user/tong/qft.html`. Thanks to everybody who helped to improve it, especially Anders Lauvland.