Introduction to quantum field theory (FYS 4170)

recall : quantization à la Schrödinger
w/ • relativistic energy-momentum relation
• number N of particles is conserved

=> 1. concept of particles ?
2. violation of causality
3. negative energy states
[4. spin ?]
5. probability interpretation !?

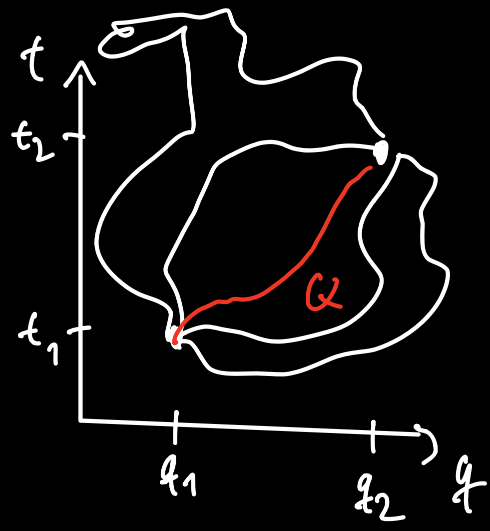
⇝ try to construct a multiparticle ("field") theory!

N not conserved

1. Classical field theory

Lagrangian and Hamiltonian

point particle in 1D : action $S = \int dt \, L[q(t), \dot{q}(t)]$



Q : physical trajectory / "classical path"

⇝ satisfies

$$\delta S \overset{!}{=} 0 \iff \boxed{\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0}$$

(⇝ solution : $q(t) = Q(t)$)

1D → N dimensions : $q \to q = (q_1, \dots, q_N)$

$$\dot{q} = \frac{d}{dt} q$$

field theory : consider a Lagrangian density instead
[required by locality!]

$$S = \int dt\, L = \int d^4x\, \mathcal{L}(\phi, \partial_\mu \phi)$$

"Lagrangian" $\phi(x^\mu)$ "field"

$\hat{=} q_i$

principle of least action :

$$0 \overset{!}{=} \delta S = \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \underbrace{\delta(\partial_\mu \phi)}_{\partial_\mu (\delta\phi)} \right\}$$

NB: sum convention!

$$= \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \underbrace{\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\delta\phi \right)} - \delta\phi\, \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \right) \right\}$$

$\checkmark$

$\rightarrow \int_{\partial V} d\Sigma_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\delta\phi$

$\rightarrow 0$ if i) $\delta\phi(t_1, \vec{x}) = \delta\phi(t_2, \vec{x}) = 0$

ii) $\delta\phi(t_1 < t < t_2, \vec{x}) \rightarrow 0$ sufficiently fast for $|\vec{x}| \to \infty$

$$\Rightarrow 0 \overset{!}{=} \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \right) \right\} \delta\phi \quad \underline{\underline{\forall\, \delta\phi(x^\mu)}}\ !$$

$$\Rightarrow \boxed{\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} - \frac{\partial \mathcal{L}}{\partial \phi} = 0}$$

"Euler Lagrange equations"

$\rightsquigarrow$ equations of motion for $\phi(x^\mu)$

NB: straightforward to describe multiple fields:

$\phi \rightarrow \phi_a$

recall: $p \equiv \frac{\partial L}{\partial \dot{q}}$ (in classical mechanics)

$\rightsquigarrow$ canonical(!) momentum <u>density</u> $\boxed{\pi(x^\mu) \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}}}$

$\Rightarrow$ "Hamiltonian" $\boxed{H \equiv \int d^3x \, \underbrace{\{ \pi(x^\mu) \dot{\phi}(x^\mu) - \mathcal{L} \}}_{\equiv \mathcal{H} = \mathcal{H}(\phi, \pi, \vec{\nabla}\phi)}}$

<u>example</u> : scalar field $\phi(x^\mu)$

$$\Rightarrow \mathcal{L} = \frac{1}{2} \underbrace{(\partial_\mu \phi)^2}_{(\partial_\mu \phi)(\partial^\mu \phi)} - V(\phi) \rightarrow \frac{1}{2} m^2 \phi^2 = \text{" } 1^{st} \text{ term in Taylor expansion"}$$

$$\text{"="} \quad T - V$$

$$\Rightarrow \cdot \frac{\partial \mathcal{L}}{\partial \phi} = -V' \rightarrow -m^2 \phi$$

$$\cdot \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \frac{1}{2} \frac{\partial}{\partial(\partial_\mu \phi)} \left[(\partial_\nu \phi)(\partial_\sigma \phi) \eta^{\nu\sigma} \right]$$

$$\eta^{\nu\sigma} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$= \frac{1}{2} \left[\underbrace{\delta^\mu_\nu (\partial_\sigma \phi) \eta^{\nu\sigma} + (\partial_\nu \phi) \delta^\mu_\sigma \eta^{\nu\sigma}}_{\partial^\mu \phi} \right]$$

$$= \partial^\mu \phi$$

(e.o.m.) $\Rightarrow$ $\boxed{(\underbrace{\partial^\mu \partial_\mu}_{= \Box} + m^2)\phi = 0}$ "Klein-Gordon equation" (for a <u>classical</u> field)

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi} \Rightarrow \mathcal{H} = \pi \dot{\phi} - \mathcal{L}$$

$$= \underbrace{\tfrac{1}{2}\dot{\phi}^2}_{\text{"kinetic energy"}} + \underbrace{\tfrac{1}{2}(\vec{\nabla}\phi)^2}_{\text{"shearing in space"}} + \underbrace{\tfrac{1}{2}m^2\phi^2}_{\text{"having the field around at all"}} > 0 \checkmark$$

Noether's theorem

"for every symmetry there is a conservation law"

consider a continuous field transformation of a physical field

$$(*) \quad \phi(x) \longrightarrow \phi'(x) \equiv \phi(x) + (\alpha)\Delta\phi(x)$$

α: small parameter

$$\Rightarrow \mathcal{L} \rightarrow \mathcal{L}' = \mathcal{L} + (\alpha)\Delta\mathcal{L}$$

(*) is called a "symmetry" iff the equations of motion do not change under (*)

- sufficient condition : $\Delta\mathcal{L} = \partial_\mu J^\mu(x)$

[not using equations of motion]

• in general : $\mathcal{L} \to \mathcal{L}' = \mathcal{L}(\phi', \partial_\mu \phi')$

[using e.o.m.]

$$= \mathcal{L}(\phi, \partial_\mu \phi) + \underbrace{\alpha \frac{\partial \mathcal{L}}{\partial \phi} \Delta\phi + \alpha \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \overbrace{\Delta \partial_\mu \phi}^{\partial_\mu \Delta\phi}}_{= \alpha \Delta \mathcal{L}} + O(\alpha^2)$$

$$= \alpha \left\{ \underbrace{\left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \right)}_{=0 \text{ (e.o.m.) !}} \Delta\phi + \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi \right\}$$

⇒ if there is a symmetry (i.e. J^μ exists), then

$$\boxed{\partial_\mu \underbrace{\left(J^\mu - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi \right)}_{\equiv j_N^\mu \text{ "Noether current"}} = 0}$$

⇒ "For each symmetry $\Delta\phi$, there is a conserved current j_N^μ (and conserved charge $Q = \int d^3x \, j_N^0$)"

application : the energy-momentum tensor

space-time translations : $x^\nu \longrightarrow x'^\nu = x^\nu + a^\nu$

NB: 4 different symmetries ($\nu = 0, 1, 2, 3$)

$$\Rightarrow \phi(x) \longrightarrow \phi'(x) = \phi(x+a) = \phi(x) + a^\nu \underbrace{\partial_\nu \phi(x)}_{\equiv (\Delta\phi)_\nu} + \mathcal{O}(a^2)$$

$$\Rightarrow \mathcal{L} \rightarrow \mathcal{L} + a^\nu \partial_\nu \mathcal{L} = \mathcal{L} + a^\nu \partial_\mu \underbrace{(\delta^\mu_\nu \mathcal{L})}_{\equiv (\mathcal{J}^\mu)_\nu}$$

$\Rightarrow$ four conserved currents

$$(j^\mu)_\nu \equiv \boxed{T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \mathcal{L} \delta^\mu_\nu}$$

$\Rightarrow$ conserved charges : i) [$\nu=0$] $H = \int T^{00} d^3x = \int \mathcal{H} d^3x$

ii) [$\nu=i$] $P^i = \int T^{0i} d^3x = -\int \pi \partial_i \phi \, d^3x$

$\rightsquigarrow$ physical momentum!

($\Leftrightarrow$ conjugate/canonical momentum π)