The Lorentz group

consider $x^\mu \longrightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu$

Λ is a Lorentz transformation

$\Leftrightarrow \; x^2 \equiv \eta_{\mu\nu} x^\mu x^\nu \overset{!}{=} x'^2$

$\Leftrightarrow \; x^T \eta x \overset{!}{=} (x')^T \eta x' = (\Lambda x)^T \eta (\Lambda x)$

$= x^T \Lambda^T \eta \Lambda x$

$\Leftrightarrow \; \boxed{\eta = \Lambda^T \eta \Lambda}$ (*)

$\left(\Leftrightarrow \; \eta_{\mu\nu} = \Lambda^\tau{}_\mu \eta_{\tau\sigma} \Lambda^\sigma{}_\nu \right)$

c.f. 3D rotations : $x^i \longrightarrow R^{ij} x^j$

R is a rotation $\Leftrightarrow x^2 \equiv \vec{x} \cdot \vec{x} = \text{const.}$

$\Leftrightarrow \; \mathbb{1}_{3\times3} = R^T \mathbb{1}_{3\times3} R$

$\Leftrightarrow$ "R is orthogonal"

- LTs form a group L : $\{\{\Lambda\}, \cdot\}$
  - $\Lambda_1, \Lambda_2 \in L \Rightarrow \Lambda_1 \cdot \Lambda_2 \in L$ "closure"
  - $\forall \Lambda_1, \Lambda_2, \Lambda_3 \in L : (\Lambda_1 \cdot \Lambda_2) \cdot \Lambda_3 = \Lambda_1 \cdot (\Lambda_2 \cdot \Lambda_3)$ "associativity"

• $\forall \Lambda \in L : \underset{\substack{\uparrow \\ L}}{\mathbb{1}} \Lambda = \Lambda \cdot \mathbb{1} = \Lambda$ "identity"

• $\forall \Lambda \in L \; \exists \Lambda^{-1} \in L : \Lambda \cdot \Lambda^{-1} = \Lambda^{-1} \Lambda = \mathbb{1}$ "inverse"

but <u>not</u> Abelian, i.e. in general $\Lambda_1 \cdot \Lambda_2 \neq \Lambda_2 \cdot \Lambda_1$

<u>decomposition of Lorentz group L</u>

$(*) \Rightarrow$ i) $\det \eta = \det \Lambda \cdot \det \eta \cdot \det \Lambda$

$\Rightarrow \det \Lambda = \pm 1$

ii) $\eta_{00} = 1 = \eta_{\rho\sigma} \Lambda^{\rho}{}_{0} \Lambda^{\sigma}{}_{0} = (\Lambda^0{}_0)^2 - (\Lambda^1{}_0)^2 - (\Lambda^2{}_0)^2 - (\Lambda^3{}_0)^2$

$\Rightarrow (\Lambda^0{}_0)^2 \geq 1$

$\Rightarrow \Lambda^0{}_0 \begin{matrix} \geq 1 \\ \leq -1 \end{matrix}$

$\Rightarrow L$ splits into 4 disconnected subsets:

$L = L_+^{\uparrow} \cup L_+^{\downarrow} \cup L_-^{\uparrow} \cup L_-^{\downarrow}$ $\quad \pm : \det \Lambda = \pm 1$ $\quad \uparrow\downarrow : \Lambda^0{}_0 \begin{matrix} \geq +1 \\ \leq -1 \end{matrix}$

group of proper Lorentz trafos

$\Rightarrow$ general LT : $\boxed{\Lambda = P^m T^n \Lambda_0}$; $m, n \in \{0,1\}$

where $\Lambda_0 \in L_+^{\uparrow}$: boosts and rotations

$P : \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ "spatial reflection"

$T : \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ "temporal reflection"

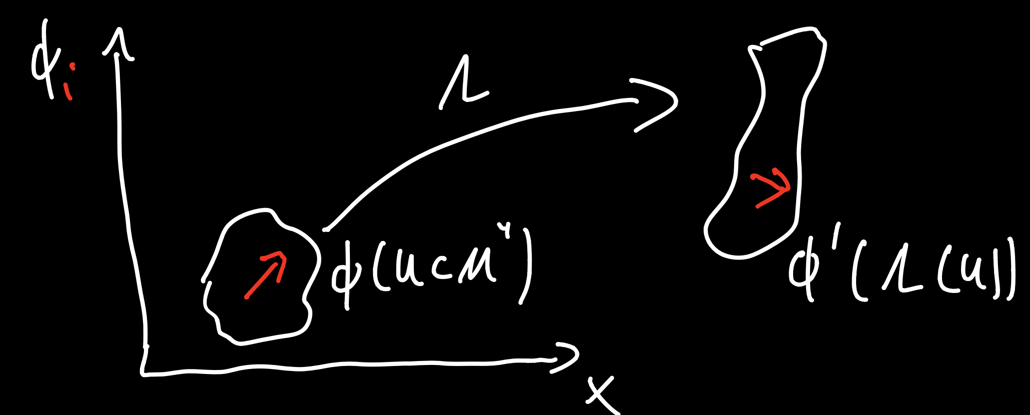
<u>relativistic invariance</u>

an expression is "relativistically invariant" if it takes the same form

a) in all frames of reference ("passive" point of view)

b) after boosting/rotating all fields ("active" —)

$\rightsquigarrow$ both are equivalent! we will adopt this one !



$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

$$\Rightarrow \boxed{\begin{aligned} \phi(x) &\to \phi'(x) = \phi(\Lambda^{-1}x) \\ V^\mu(x) &\to V'^\mu(x) = \Lambda^\mu{}_\nu V^\nu(\Lambda^{-1}x) \end{aligned}}$$

"scalar" field

"4-vector" field

NB: $\mathcal{L}$ is a Lorentz scalar

$\Rightarrow$ equations of motion are automatically relativistically invariant!

The Lorentz algebra

motivation: how to construct Lorentz-invariant equations?

1st guess: "count indices" [each term must have the same set of uncontracted indices]

$\rightsquigarrow$ problem: this gives only a subset of all possibilities!

more general: find <u>all</u> possible transformation laws for an N-component field $\phi_a(x)$

$a = 1 \dots N$; <u>not</u> a space-time index

<u>Solution</u> : • let's restrict ourselves to infinitesimal transformations [$\mathcal{L}^\uparrow_+$ is a continuous group!]

$$\Rightarrow \phi_a(x) \xrightarrow{LT} \phi'_a = M_{ab}(\Lambda)\,\phi_b(\Lambda^{-1}x)$$

$N \times N$ matrix

• only requirement on M:
preserve correspondence between M & Λ for subsequent LT's
(i.e. M must be a <u>representation</u> of $\mathcal{L}$!)

$$\Leftrightarrow \boxed{\Lambda'' = \Lambda'\Lambda \Rightarrow M_{ab}(\Lambda'') = M_{ac}(\Lambda')\,M_{cb}(\Lambda)} \quad (*)$$

$\rightarrow$ now find <u>all</u> solutions to this!

i) we consider <u>infinitesimal</u> transformations:

$$\boxed{\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu\,, \quad |\omega^\mu{}_\nu| \ll 1\,, \quad \omega_{\mu\nu} \overset{(*)}{=} -\omega_{\nu\mu}}$$

$$\Rightarrow \boxed{M_{ab}(\Lambda) = \delta_{ab} - \underbrace{\tfrac{i}{2}}_{\text{(convention!)}}\omega_{\mu\nu}(J^{\mu\nu})_{ab} + \dots\,; \quad J^{\mu\nu} = -J^{\nu\mu}}$$

⌈ why $\omega_{\mu\nu} = -\omega_{\nu\mu}$? expand $\eta = \Lambda^T \eta \Lambda$!

$$\Leftrightarrow \eta_{\mu\nu} = \eta_{\rho\sigma}(\delta^\rho_\mu + \omega^\rho{}_\mu)(\delta^\sigma_\nu + \omega^\sigma{}_\nu)$$

$$= \eta_{\mu\nu} + \omega_{\mu\nu} + \omega_{\nu\mu} + \mathcal{O}(\omega^2)$$

□ ⌋

ii) apply (*) to $\Lambda'' = \Lambda \Lambda' \Lambda^{-1}$

⌈ inf. Λ'

$$\Rightarrow M(\Lambda(\mathbb{1}+\omega')\Lambda^{-1}) \overset{!}{=} M(\Lambda)\, M(\Lambda' = \mathbb{1}+\omega')\, \underbrace{M(\Lambda^{-1})}_{= M^{-1}(\Lambda)}$$

only $\mathcal{O}(\omega')$

$$\Leftrightarrow \frac{i}{2} \underbrace{(\Lambda \omega' \Lambda^{-1})_{\mu\nu}}_{\Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma \omega'_{\rho\sigma}} J^{\mu\nu} = M(\Lambda)\left(\frac{i}{2}\omega'_{\mu\nu} J^{\mu\nu}\right) M^{-1}(\Lambda)$$

| $\eta = \Lambda^T \eta \Lambda$

$\Rightarrow \eta \Lambda^{-1} = \Lambda^T \eta$

$\Rightarrow \Lambda^{-1} \,{}^{"}{=}^{"}\, \Lambda^T$

$$\Leftrightarrow \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma J^{\mu\nu} = M(\Lambda) J^{\rho\sigma} M^{-1}(\Lambda)$$

inf $\Lambda = \mathbb{1}+\omega$

$$\Leftrightarrow \underbrace{\omega^\rho{}_\mu J^{\mu\sigma} + \omega_\nu{}^\sigma J^{\rho\nu}}_{} = \left(\frac{i}{2}\omega_{\mu\nu} J^{\mu\nu}\right) J^{\rho\sigma} \underset{M^{-1}}{\ominus} J^{\rho\sigma}\left(\frac{i}{2}\omega_{\mu\nu} J^{\mu\nu}\right)$$

$$= \omega_{\mu\nu}\{\eta^{\nu\rho} J^{\mu\sigma} \underset{\omega_{\nu\mu}}{\ominus} \eta^{\mu\sigma} J^{\rho\nu}\} \quad | \text{ use that } J^{\mu\nu} = -J^{\nu\mu} ! ⌋$$

"Lorentz algebra"

$$\Leftrightarrow \boxed{-i[J^{\mu\nu}, J^{\rho\sigma}] = \eta^{\nu\sigma} J^{\rho\mu} - \eta^{\mu\sigma} J^{\rho\nu} + \eta^{\nu\rho} J^{\mu\sigma} - \eta^{\mu\rho} J^{\nu\sigma}}$$

$\to$ 6 "generators" $J^{\mu\nu}$ $(= -J^{\nu\mu})$

$\overset{\uparrow}{}$ $N \times N$ matrices

$\shortparallel$

$3+3$!

"boosts" + "rotations"

<u>examples</u> : 1) $J^{\mu\nu} \equiv x^\mu \hat{p}^\nu - x^\nu \hat{p}^\mu$

$$= i(x^\mu \partial^\nu - x^\nu \partial^\mu) \qquad \Big| \ \partial_\mu = (\partial_t, \vec{\nabla}); \partial^\mu = (\partial_t, -\vec{\nabla})$$

in 3D : $J^{ij} = -i(x^i \nabla^j - x^j \nabla^i)$ $\quad \Big| \ J^1 \equiv J^{23}; J^2 \equiv J^{31}; J^3 \equiv J^{12}$

$$\Leftrightarrow J^i = \varepsilon^{ijk} x^j (-i\nabla^k)$$

$$\Leftrightarrow \vec{J} = \vec{x} \times \vec{p}$$

$$\Rightarrow [J^i, J^j] = i\varepsilon^{ijk} J^k$$

- from QM : angular momentum operators
- today : these J^i form a subset of the Lorentz algebra !

2) Consider now 4×4 matrices $\tilde{J}^{\mu\nu}$, with

$$\boxed{(\tilde{J}^{\mu\nu})_{\alpha\beta} \equiv i(\delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha)}$$

$\Rightarrow$ these are the matrices that generate Lorentz transformations acting on ordinary 4-vectors !

• infinitesimal: $V^\alpha \longrightarrow \Big(\delta^\alpha_{\ \beta} \underbrace{- \frac{i}{2} \tilde{\omega}_{\mu\nu} (\tilde{J}^{\mu\nu})^\alpha_{\ \beta}}_{\equiv \omega^\alpha_{\ \beta}} \Big) V^\beta$

• finite: $V^\alpha \longrightarrow \Lambda^\alpha_{\ \beta} V^\beta$;

$$\boxed{\Lambda^\alpha_{\ \beta} = e^{-\frac{i}{2}\tilde{\omega}_{\mu\nu}\tilde{J}^{\mu\nu}}}$$

recall $\exp[x] = \lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n$

e.g. • $\tilde{\omega}_{12} = -\tilde{\omega}_{21} \equiv \theta$ (all remaining $\omega_{\mu\nu} = 0$)

$$\Rightarrow (\tilde{J}^{12})^\alpha_{\ \beta} = \eta^{\alpha\gamma} (\tilde{J}^{12})_{\gamma\beta} = -i \begin{pmatrix} 0&0&0&0 \\ 0&0&-1&0 \\ 0&+1&0&0 \\ 0&0&0&0 \end{pmatrix} = -(\tilde{J}^{21})^\alpha_{\ \beta}$$

$$\Rightarrow V \to \begin{pmatrix} 1&0&0&0 \\ 0&1&-\theta&0 \\ 0&\theta&1&0 \\ 0&0&0&1 \end{pmatrix} V$$: inf. rotation around z-axis !

⌜exercise: finite θ⌟

• $\tilde{\omega}_{01} = -\tilde{\omega}_{10} \equiv \eta$ "rapidity"

$$\Rightarrow V \to \begin{pmatrix} 1&\beta&0&0 \\ \beta&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix}$$: inf. boost in x direction!

⌜exercises: derive finite form⌟