2. The Klein-Jordon field

classical real scalar field (free)

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 \Rightarrow$$

- $(\partial^2 + m^2)\phi = 0$ KGE
- $\mathcal{H} = \pi\dot{\phi} - \mathcal{L}$ $\quad | \; \pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}$

$$= \frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2$$

quantization

QM : q_i, p_i $\longrightarrow$ operators $\hat{q}_i, \hat{p}_i$

classical coordinates / phase-space variables

with $[\hat{q}_i, \hat{p}_j] = i\delta_{ij}$

$[\hat{q}_i, \hat{q}_j] = [\hat{p}_i, \hat{p}_j] = 0$

NB: Schrödinger picture

$\to$ no t-dependence of operators

QFT : ϕ, π $\longrightarrow$ $\overset{(\wedge)}{\phi}, \overset{(\wedge)}{\pi}$

classical fields

at some fixed value $t = t_0$

with $\boxed{\begin{aligned} [\phi(\vec{x}), \pi(\vec{y})] &= i\,\delta^{(3)}(\vec{x} - \vec{y}) \\ [\phi(\vec{x}), \phi(\vec{y})] &= [\pi(\vec{x}), \pi(\vec{y})] = 0 \end{aligned}}$

"equal time commutation relations"

energy spectrum

Fourier transform only w.r.t. $\vec{x}$ ("keep t fixed"/ t-dependence explicit)

$$\phi([t],\vec{x}) = \int \frac{d^3p}{(2\pi)} e^{i\vec{p}\vec{x}} \phi([t,]\vec{p})$$

$$\Rightarrow \text{KGE}: \boxed{\Big(\frac{\partial^2}{\partial t^2} + \underbrace{\vec{p}^2 + m}_{\equiv \omega_p^2}\Big)\phi = 0}$$

$\rightsquigarrow$ harmonic oscillator! ($\rightsquigarrow \phi \sim e^{\pm i\omega_p t}$)

⌈ recall from QM: $H_{SHO} = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 x^2$ | $x \equiv \frac{1}{\sqrt{2\omega}}(a+a^+)$

$= \omega(a^+a + \frac{1}{2})$ $\qquad p \equiv -i\sqrt{\frac{\omega}{2}}(a-a^+)$

$\overset{[x,p]}{\Longrightarrow} [a,a^+] = 1$ ⌋

define "ladder operators"

$$\hat{\phi}(t,\vec{x}) \equiv \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(\hat{a}_{\vec{p}}\, e^{i\vec{p}\vec{x}} + a^{\dagger}_{\vec{p}}\, e^{-i\vec{p}\vec{x}}\right)$$

$$= \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\vec{x}} \underbrace{\frac{1}{\sqrt{2\omega_p}}\left(\hat{a}_{\vec{p}} + a^{\dagger}_{-\vec{p}}\right)}_{\phi(t,\vec{p})}$$

$$\pi(t,\vec{x}) \overset{(\dot{\phi})}{\equiv} \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\vec{x}} (-i)\sqrt{\frac{\omega_p}{2}} \left(a_{\vec{p}} - a^{\dagger}_{-\vec{p}}\right)$$

$$\Leftrightarrow \quad a_{\vec{p}} = \sqrt{\frac{\omega_p}{2}} \left(\phi(\vec{p}) + \frac{i}{\omega_p} \pi(\vec{p}) \right)$$

$$a^{\dagger}_{\vec{p}} = \sqrt{\frac{\omega_p}{2}} \left(\phi(-\vec{p}) - \frac{i}{\omega_p} \pi(-\vec{p}) \right)$$

$$\Rightarrow [a_p, a^{\dagger}_{\vec{p}'}] = \frac{1}{2}\sqrt{\omega_p \omega_{p'}} \left[\phi(\vec{p}) + \frac{i}{\omega_p}\pi(\vec{p}), \phi(-\vec{p}') - \frac{i}{\omega_{p'}}\pi(-\vec{p}') \right]$$

$$= \frac{1}{2}\sqrt{\omega_p \omega_{p'}} \int d^3x \int d^3y \, e^{-i\vec{p}\vec{x}} \, e^{+i\vec{p}'\vec{y}} \times$$

$$\times \underbrace{\left[\phi(\vec{x}) + \frac{i}{\omega_p}\pi(\vec{x}), \phi(\vec{y}) - \frac{i}{\omega_{p'}}\pi(\vec{y}) \right]}_{= -\frac{i}{\omega_{p'}}[\phi(\vec{x}), \pi(\vec{y})] + \frac{i}{\omega_p} \underbrace{[\pi(\vec{x}), \phi(\vec{y})]}_{-[\phi(\vec{y}), \pi(\vec{x})]}}$$

$$= \delta^{(3)}(\vec{x}-\vec{y}) \left(\frac{1}{\omega_p} + \frac{1}{\omega_{p'}} \right)$$

$$= \frac{1}{2}\sqrt{\omega_p \omega_{p'}} \underbrace{\int d^3x \, e^{-i\vec{x}(\vec{p}-\vec{p}')}}_{(2\pi)^3 \delta^{(3)}(\vec{p}-\vec{p}') \Rightarrow \omega_p = \omega_{p'}} \left(\frac{1}{\omega_p} + \frac{1}{\omega_{p'}} \right)$$

$$\Rightarrow \boxed{[a_{\vec{p}}, a^{\dagger}_{\vec{p}'}] = (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{p}')}$$

similar: $[a_{\vec{p}}, a_{\vec{p}'}] = [a^{\dagger}_{\vec{p}}, a^{\dagger}_{\vec{p}'}] = 0$

$$\Rightarrow H = \int d^3x \left\{ \frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2 \right\}$$

$$= \underline{\int d^3x} \int \frac{d^3p\, d^3p'}{(2\pi)^6} \underbrace{e^{i\vec{x}(\vec{p}+\vec{p}')}}_{\to (2\pi)^3 \delta^{(3)}(\vec{p}+\vec{p}')}$$

$$\times \left\{ -\frac{\sqrt{\omega_{\vec{p}}\omega_{\vec{p}'}}}{4}(a_{\vec{p}} - a^\dagger_{-\vec{p}})(a_{\vec{p}'} - a^\dagger_{-\vec{p}'}) + \frac{-\vec{p}\cdot\vec{p}' + m^2}{4\sqrt{\omega_{\vec{p}}\omega_{\vec{p}'}}}(a_{\vec{p}} + a^\dagger_{-\vec{p}})(a_{\vec{p}'} + a^\dagger_{-\vec{p}'}) \right\}$$

$$= \int \frac{d^3p}{(2\pi)^3} \left\{ -\frac{\omega_p}{4}(a_{\vec{p}} - a^\dagger_{-\vec{p}})(a_{-\vec{p}} - a^\dagger_{\vec{p}}) + \underbrace{\frac{\vec{p}^{\,2} + m^2}{4\omega_p}}_{\frac{\omega_p}{4}}(a_{\vec{p}} + a^\dagger_{-\vec{p}})(a_{-\vec{p}} + a^\dagger_{\vec{p}}) \right\}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{\omega_p}{4} 2 \left\{ a_{\vec{p}} a^\dagger_{\vec{p}} + a^\dagger_{-\vec{p}} a_{-\vec{p}} \right\}$$

$-\vec{p} \to \vec{p}$ $\quad$ ($\int d^3p \to \int d^3\tilde{p}$; $\tilde{p} = -p$)

$$= \int \frac{d^3p}{(2\pi)^3} \omega_{\vec{p}} \left\{ a^\dagger_{\vec{p}} a_{\vec{p}} + \frac{1}{2} \underbrace{[a_{\vec{p}}, a^\dagger_{\vec{p}}]}_{\propto \delta^{(3)}(0) = \infty} \right\}$$

= sum over all zero-point energies $\frac{\omega_{\vec{p}}}{2}$

<u>BUT</u>: experimentally we only measure <u>differences</u> to ground state energy!

$\rightsquigarrow$ ignore ... [NB: not possible in GR ... !]

similar : $\underline{P} = -\int d^3x\, \pi(\vec{x}) \nabla \phi(\vec{x}) = \ldots = \int \frac{d^3p}{(2\pi)^3} \vec{p}\, a^\dagger_{\vec{p}} a_{\vec{p}}$

Def. <u>vacuum</u> :
- $\langle 0|0\rangle = 1$
- $a_{\vec{p}} |0\rangle = 0 \quad \forall \vec{p}$

$\Rightarrow H|0\rangle = 0$; i.e $E = 0$

<u>particle interpretation</u>

$[\underset{\underline{P}}{H}, a^\dagger_{\vec{p}}] = \underset{\vec{p}}{\omega_{\vec{p}}}\, a^\dagger_{\vec{p}}$

$[\underset{\underline{P}}{H}, a_{\vec{p}}] = -\underset{\vec{p}}{\omega_{\vec{p}}}\, a_{\vec{p}}$

$\Rightarrow \ldots$

all energy eigenstates can be written as

$a^\dagger_{\vec{p}} \ldots a^\dagger_{\vec{q}} |0\rangle$

with energy $E = \omega_{\vec{p}} + \ldots + \omega_{\vec{q}}$

and momentum $\underline{P} = \vec{p} + \ldots + \vec{q}$

$\Rightarrow a^\dagger_{\vec{p}}$ creates an excitation with • energy $\omega_{\vec{p}} = +\sqrt{\vec{p}^2 + m^2}$ > 0 !

• momentum $\vec{p}$

$\rightsquigarrow$ "particles" ! (NB: discrete, but not necessarily localized in space!)

- statistics : i) $a^\dagger_{\vec p}\, a^\dagger_{\vec q}\, |0\rangle = +a^\dagger_{\vec q}\, a^\dagger_{\vec p}\, |0\rangle$

  ii) $(a^\dagger_{\vec p})^n |0\rangle \neq 0 \quad \forall\, n \geq 0$

  $\Rightarrow$ Klein-Gordon particles obey Bose-Einstein statistics!

- conventions : $|\vec p\rangle \equiv \sqrt{2E_{\vec p}}\; a^\dagger_{\vec p}\, |0\rangle$

  $\Rightarrow \langle \vec q\,|\,\vec p\rangle = 2E_{\vec p}\,(2\pi)^3\,\delta^{(3)}(\vec p - \vec q\,)$

  $=$ Lorentz invariant!

  $$\left[\int \frac{d^4p}{(2\pi)^4}\, \underbrace{\delta(p^2 - m^2)}_{\delta((p_0+E_p)(p_0-E_p))}\, \Theta(p^0) = \int \frac{d^3p}{(2\pi)^3}\, \frac{1}{2E_p}\right]$$

  $\rightsquigarrow$ see P&S, p 22/23

- interpretation $\phi(\vec x)\,|0\rangle$

  i) $$\phi(\vec x)|0\rangle = \int \frac{d^3p}{(2\pi)^3}\, \underbrace{\frac{1}{2E_{\vec p}}}_{\to 1 \text{ in NR limit}}\, e^{-i\vec p\vec x}\,|\vec p\rangle = \int \frac{d^4p}{(2\pi)^4}\,\delta(p^2-m^2)\,\Theta(p^0)\; e^{-i\vec p\vec x}\,|\vec p\rangle$$

$\Rightarrow$ recover NR / QM expression for $|x\rangle$

ii) $$\langle 0|\phi(x)|p\rangle = \int \frac{d^3p'}{(2\pi)^3} \frac{1}{\sqrt{2E_{p'}}} \langle 0| \underbrace{(a_{\vec{p}'} + \cancel{a^\dagger_{-\vec{p}'}}) a^\dagger_{\vec{p}} |0\rangle}_{\to (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{p}')} \sqrt{2E_{\vec{p}}}\, e^{i\vec{p}x}$$

$$= e^{i\vec{p}\vec{x}}$$

$\propto \langle x|p\rangle$ in NR QM

$\Rightarrow$ "$\phi(\vec{x})$ creates a particle at position $\vec{x}$"