time dependence: from Schrödinger to Heisenberg

$$\mathcal{O}_H = e^{iHt}\,\mathcal{O}_S\,e^{-iHt}$$

$\mathcal{O}_H \leftarrow (t,\vec{x})$ $\quad$ $\mathcal{O}_S \leftarrow (\vec{x})$

$$\phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\left(a_{\vec{p}}\,e^{i\vec{p}\vec{x}} + a^\dagger_{\vec{p}}\,e^{-i\vec{p}\vec{x}}\right)$$

$$\Rightarrow \phi(x) = \phi(t,\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}\left\{\underbrace{e^{iHt}a_{\vec{p}}\,e^{-iHt}}_{a_{\vec{p}}\,e^{-iE_{\vec{p}}t}}\,e^{i\vec{p}x} + \underbrace{e^{iHt}a^\dagger_{\vec{p}}\,e^{-iHt}}_{a^\dagger_{\vec{p}}\,e^{+iE_{\vec{p}}t}}\,e^{-i\vec{p}\vec{x}}\right\}$$

$$\left[\; e^{iHt}a_{\vec{p}} = \sum_n (iHt)^n a_{\vec{p}} \quad \Big|\; [H, a_{\vec{p}}] = -\omega_p a_{\vec{p}}\right.$$

$$= \sum_n a_{\vec{p}}\left(it(H-\omega_p)\right)^n$$

$$\left. = a_{\vec{p}}\,e^{it(H-\omega_p)} \;\right]$$

$$\Rightarrow \boxed{\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left\{a_{\vec{p}}\,e^{-ip\cdot x} + a^\dagger_{\vec{p}}\,e^{+ip\cdot x}\right\}_{p^0 = \omega_p > 0}}$$

"positive frequency mode" $\qquad$ "negative frequency mode"

NB: • **inherent duality**: $a, a^\dagger$ – particle interpretation
(= quanta of field excitation)

$e^{\pm ipx}$ – wave interpretation

⤳ solutions of KG eq.

- 2 solutions for relativistic wave equation:
  - coefficient of pos. frequency mode destroys a particle w. positive energy
  - " " " neg. " " creates " " positive energy

particle propagation

amplitude for a particle from y to x:

$$D(x-y) \equiv \langle 0|\phi(x)\phi(y)|0\rangle$$

$$= \langle 0| \int \frac{d^3p\, d^3p'}{(2\pi)^6} \frac{1}{2\sqrt{E_p E_{p'}}} \left\{ a_{\vec{p}} e^{-ip\cdot x} + a_{\vec{p}}^\dagger e^{+ip\cdot x} \right\} \left\{ a_{\vec{p}'} e^{-ip'y} + a_{\vec{p}'}^\dagger e^{ip'y} \right\} |0\rangle$$

$$= \int \frac{d^3p\, d^3p'}{(2\pi)^6} \frac{1}{2\sqrt{E_p E_{p'}}} e^{-ipx+ip'y} \underbrace{\langle 0| a_p a_{p'}^\dagger |0\rangle}_{(2\pi)^3 \delta^{(3)}(\vec{p}-\vec{p}')}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip\cdot(x-y)}$$

→ does not vanish for $(x-y)^2 < 0$, [see PS, p. 25ff]
i.e. outside the light cone!?

BUT: only need to require this of observables!

e.g. measurement of $\phi(x)$ and $\phi(y)$

$\rightsquigarrow$ need to consider $[\phi(x), \phi(y)]$!

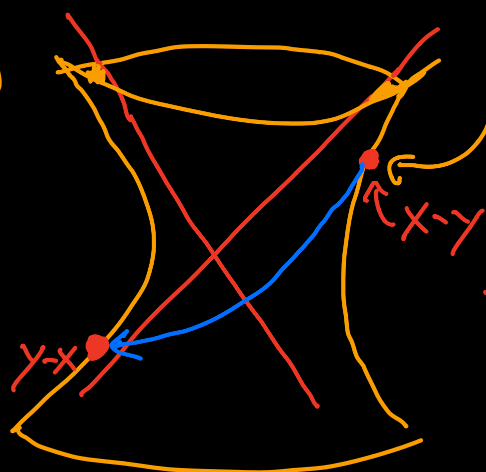
($=0$ iff the two measurements do not affect each other)

$$[\phi(x), \phi(y)] = \ldots = D(x-y) - D(y-x)$$

[NB: no $\langle 0 | \ldots | 0 \rangle$!]

$$= \begin{cases} 0 & \text{for } (x-y)^2 < 0 \\ \neq 0 & \text{for } (x-y)^2 > 0 \end{cases} \quad \checkmark$$

a)



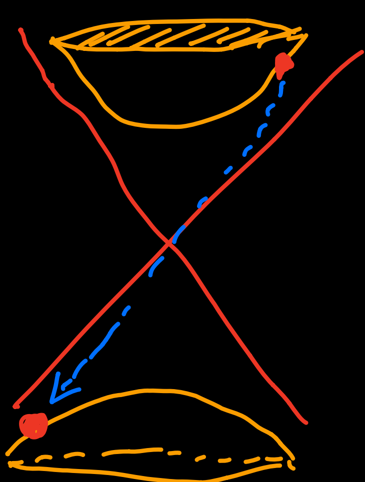
$\partial V: (x-y)^2 = \text{const.} < 0$

$\exists$ Lorentz transformation

$x-y \longrightarrow -(x-y)$

$\Rightarrow D(x-y) = D(y-x)$!

b)



$\nexists$ (cont.) Lorentz trafo

$x-y \dashrightarrow -(x-y)$

$\Rightarrow D(x-y) \neq D(y-x)$

Green's functions of Klein-Gordon operator

$$(\partial_x^2 + m^2)\, G(x-y) = -i\,\delta^{(4)}(x-y)$$

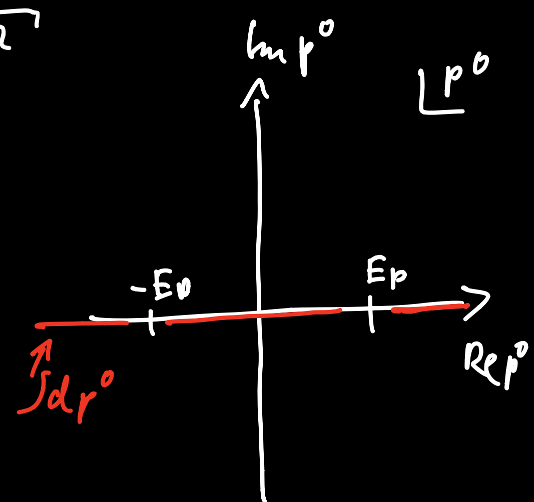
$$\int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} G(p) \qquad\qquad \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)}$$

$$(-p^2+m^2)\, G(p) = -i$$

$$\Rightarrow G(p) = \frac{i}{p^2-m^2}$$

$$\Rightarrow \boxed{G(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2} e^{-ip(x-y)}}$$

poles at $p^0 = \pm E_p = \pm\sqrt{\vec{p}^{\,2}+m^2}$

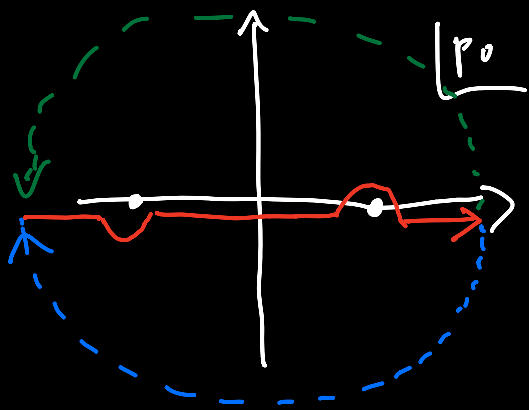


$\rightarrow$ 4 ways of treating the poles,
i.e. 4 different Green's functions

a) Feynman propagator

$$D_F(x-y) \equiv \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2+i\epsilon} e^{-ip\cdot(x-y)}$$

$\hookrightarrow (p^0-E_p)(p^0+E_p)+i\epsilon \Rightarrow p^0 = \pm(E_p - i\epsilon)$



$\Rightarrow$ i) $x^0 > y^0$ $\Rightarrow$ contours can be closed below

$\Gamma e^{-ip^0(x^0-y^0)} \longrightarrow 0$ for $p^0 \to -i\infty$

$\Rightarrow$ pick up pole at $p^0 = +E_p$

$$\Rightarrow D_F(x-y) = \int \frac{d^3p}{(2\pi)^3} \oint \frac{dp^0}{2\pi} \frac{i}{p^0-E_p} \frac{e^{-ip(x-y)}}{p^0+E_p}$$

$\oint f(z)\,dz = 2\pi i \, \mathrm{Res} f(z_0)$

$p^0 = E_p$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip\cdot(x-y)} = D(x-y) \quad !$$

ii) $x^0 < y^0$: close the contour above

$\Rightarrow e^{-ip^0(x^0-y^0)} \to 0$ for $p^0 \to +i\infty$

$\Rightarrow$ pick up pole at $p^0 = -E_p$

$$\Rightarrow D_F(x-y) = \int \frac{d^3p}{(2\pi)} \oint \frac{dp^0}{2\pi i} \frac{-1}{p^0+E_p} \frac{e^{-ip\cdot(x-y)}}{p^0-E_p}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{+ip^0(x^0-y^0)+i\vec{p}\cdot(\vec{x}-\vec{y})}$$

$-$ after $\vec{p} \to -\vec{p}$

$$= D(y-x)$$

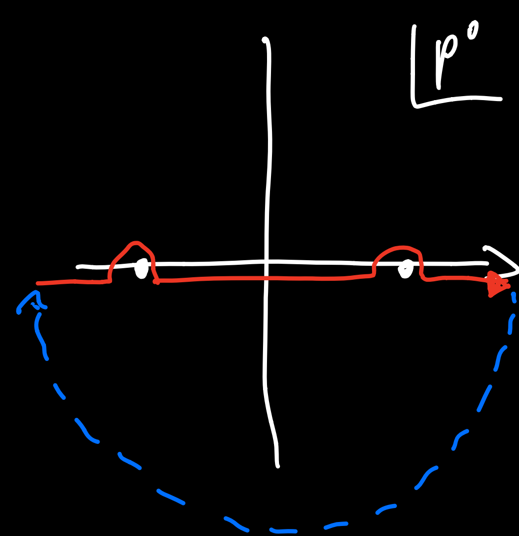
$$\Rightarrow \boxed{D_F(x-y) = \begin{cases} D(x-y) & \text{for } x^0 > y^0 \\ D(y-x) & \text{for } y^0 > x^0 \end{cases} \equiv \langle 0 | T \phi(x) \phi(y) | 0 \rangle}$$

"time-ordering" T: order all operators by following the "T", latest to the left.

b) retarded Green's function

(vanishes for $x^0 < y^0$)

$\rightsquigarrow$ take a contour above both poles (need $x^0 > y^0$ to pick up both)



$$\Rightarrow D_R(x-y) = \int \frac{d^3p}{(2\pi)^3} \oint \frac{dp^0}{2\pi i} \frac{1}{p^0 - E_p} \frac{1}{p^0 + E_p} e^{-ip\cdot(x-y)}$$

$$= \Theta(x^0 - y^0) \int \frac{d^3p}{(2\pi)^3} \left\{ \frac{1}{2E_p} e^{-ip(x-y)} - \frac{1}{2E_p} e^{+ip(x-y)} \right\}$$

$$= \Theta(x^0 - y^0) \left\{ D(x-y) - D(y-x) \right\}$$

$$\Rightarrow \boxed{D_R(x-y) = \Theta(x^0 - y^0) \langle 0 | [\phi(x), \phi(y)] | 0 \rangle}$$