3. The Dirac algebra

recall Lorentz algebra:

$$(*)\quad [J^{\mu\nu}, J^{\rho\sigma}] = i\left(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}\right)$$

- goal: look for a finite-dimensional representation that corresponds to spin 1/2

⇝ "idea": take $n\times n$ matrices γ^μ with

$$\boxed{\{\gamma^\mu,\gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\times \mathbb{1}_{n\times n}} \quad (**)$$

"Dirac / Clifford algebra"

$\Rightarrow (\gamma^0)^2 = \mathbb{1}$

$(\gamma^i)^2 = -\mathbb{1}$

$$\Rightarrow \boxed{S^{\mu\nu} \equiv \frac{i}{4}[\gamma^\mu,\gamma^\nu]} \text{ satisfies } (*)\,!$$

→ exercise: show this! (warning: rather technical...)

remark: you already "know" this in 3D!

Def. $\gamma^i = i\sigma^i$ Pauli matrices: $\sigma^1 = \begin{pmatrix}0&1\\1&0\end{pmatrix}$ $\sigma^2 = \begin{pmatrix}0&-i\\i&0\end{pmatrix}$ $\sigma^3 = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$

$$\Rightarrow \{\gamma^i,\gamma^j\} = -\{\sigma^i,\sigma^j\} = -2\delta^{ij} \;\checkmark$$

[as required by (**)]

- $S^{ij} = -\frac{i}{4}[\sigma^i, \sigma^j] = \frac{1}{2}\varepsilon^{ijk}\sigma^k$ [c.f. earlier 3D discussion of (*)!]

$\Rightarrow$ Pauli matrices are a representation of the rotation group!

"the spin $\frac{1}{2}$" representation

Lorentz transformation properties

$\psi \equiv \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_n \end{pmatrix}$ is called a Dirac spinor if it transforms under Lorentz transformations with $S^{\mu\nu}$, i.e.

$$\boxed{\psi_a(x) \longrightarrow \psi'_a(x) = \underbrace{M(\Lambda)}_{\equiv (\Lambda_{\frac{1}{2}})_{ab}} \psi_b(\Lambda^{-1}x)}$$

with

$$\boxed{\begin{array}{ll} \Lambda = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\tilde{J}^{\mu\nu}\right) & i(\tilde{J}^{\mu\nu})_{ab} = i(\delta^\mu_a \delta^\nu_b - \delta^\mu_b \delta^\nu_a) \\ \Lambda_{\frac{1}{2}} = \exp\left(-\frac{i}{2}\omega_{\mu\nu} S^{\mu\nu}\right) & \end{array}}$$

- how does the γ^μ "transform"? (NB: γ^μ are constants!)

$\rightsquigarrow$ what we mean is the following:

Consider $\gamma^\mu \psi \longrightarrow \gamma^\mu \Lambda_{1/2} \psi \equiv \Lambda_{1/2} \gamma^{\mu\prime} \psi$

$\rightsquigarrow \gamma^{\mu\prime} = \Lambda_{1/2}^{-1} \gamma^\mu \Lambda_{1/2}$

$$\Big[\text{for } \omega \ll 1 : \Lambda_{1/2}^{-1} \gamma^\mu \Lambda_{1/2} = \left(\mathbb{1} + \frac{i}{2}\omega_{\rho\sigma} S^{\rho\sigma}\right) \gamma^\mu \left(\mathbb{1} - \frac{i}{2}\omega_{\rho'\sigma'} S^{\rho'\sigma'}\right)$$

$$= \gamma^\mu + \frac{i}{2}\omega_{\rho\sigma}\left[S^{\rho\sigma}, \gamma^\mu\right]$$

$$= \gamma^\mu - \frac{1}{8}\omega_{\rho\sigma}\Big\{ \underbrace{\underbrace{(\gamma^\rho\gamma^\sigma - \gamma^\sigma\gamma^\rho)}_{2(\gamma^\rho\gamma^\sigma - g^{\rho\sigma})}\gamma^\mu - \gamma^\mu \underbrace{(\gamma^\rho\gamma^\sigma - \gamma^\sigma\gamma^\rho)}_{2(\gamma^\rho\gamma^\sigma - g^{\rho\sigma})}\Big\}}_{2(\gamma^\rho\gamma^\sigma\gamma^\mu - \gamma^\mu\gamma^\rho\gamma^\sigma)}$$

$$= 4\left(g^{\sigma\mu}\gamma^\rho - g^{\mu\rho}\gamma^\sigma\right)$$

$$= \gamma^\mu - \frac{1}{2}\omega_{\rho\sigma} \underbrace{\left(g^{\mu\sigma}\delta^\rho_{\ \nu} - g^{\mu\rho}\delta^\sigma_{\ \nu}\right)}_{g^{\mu\tau}\left(\delta^\sigma_{\ \tau}\delta^\rho_{\ \nu} - \delta^\rho_{\ \tau}\delta^\sigma_{\ \nu}\right)} \gamma^\nu$$

$$= i\, g^{\mu\tau}\left(\tilde{J}^{\rho\sigma}\right)_{\tau\nu}$$

$$= \left(\mathbb{1} - \frac{i}{2}\omega_{\rho\sigma}\tilde{J}^{\rho\sigma}\right)^\mu_{\ \nu}\gamma^\nu$$

$\Rightarrow \gamma^{\mu\prime} = \boxed{\Lambda_{1/2}^{-1}\gamma^\mu\Lambda_{1/2} = \Lambda^\mu_{\ \nu}\gamma^\nu}$ i.e. $\gamma^\mu \psi$ transforms like a four vector! + spinor

Some basic facts about γ matrices

a) $\boxed{(\gamma^\mu)^\dagger = (\gamma^\mu)^{-1}}$: can be chosen unitary because they form a rep. of a finite group

⌜consider any rep. of G and a hermitian product $(\,,\,)$.

$$\rightarrow (x,y)' \equiv \sum_{g\in G} (gx, gy)$$

$$\Rightarrow \forall h \in G : (hx,y)' = \sum_g (ghx, gy)$$

$$= \sum_g (ghx, ghh^{-1}y)$$

"g'"

$$= \sum_{g'} (g'x, g'h^{-1}y)$$

$$= (x, h^{-1}y)' \quad \square$$ ⌟

b) $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \Rightarrow \bullet\ (\gamma^0)^2 = 1\ (\times \mathbb{1}_{4\times4}) \quad \Big|\ \mathbb{1}^\dagger = \mathbb{1}$

$$\Rightarrow 1 = (\gamma^{0\dagger})^2 = (\gamma^0)^\dagger (\gamma^0)^{-1}$$

$$\Rightarrow \boxed{(\gamma^0)^\dagger = \gamma^0}$$

$\bullet\ (\gamma^i)^2 = -1 \Rightarrow \ldots \Rightarrow \boxed{(\gamma^i)^\dagger = -\gamma^i}$

c) $\gamma^{\mu\dagger}\gamma^0 = \begin{cases} \gamma^0\gamma^0 & \text{for } \mu=0 \\ -\gamma^i\gamma^0 & \text{for } \mu=i \end{cases} = \boxed{\gamma^0\gamma^\mu = \gamma^{\mu\dagger}\gamma^0}$

Dirac bilinears

⇝ How to get a Lorentz scalar from ψ?

NB: generators not hermitian, i.e. $(S^{\mu\nu})^\dagger \neq S^{\mu\nu}$

$\Rightarrow \Lambda_{1/2}$ not unitary, i.e. $\Lambda_{1/2}^\dagger \neq \Lambda_{1/2}^{-1}$

$\Rightarrow \psi_a^\dagger \psi_a \xrightarrow{\text{L.T.}} \psi^\dagger \Lambda_{1/2}^\dagger \Lambda_{1/2}\psi \neq \psi^\dagger\psi$ ↯

solution: $\boxed{\bar\psi \equiv \psi^\dagger\gamma^0}$

now: $\bar\psi \longrightarrow (\Lambda_{1/2}\psi)^\dagger\gamma^0 \overset{\omega\ll 1}{=} \psi^\dagger(\mathbb{1}+\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu\dagger})\gamma^0$

$$\left| \begin{aligned} S^{\mu\nu\dagger} &= -\frac{i}{4}[\gamma^\mu,\gamma^\nu]^\dagger \\ &= \frac{i}{4}[\gamma^{\mu\dagger},\gamma^{\nu\dagger}] \\ \Rightarrow S^{\mu\nu\dagger}\gamma^0 &= \frac{i}{4}\gamma^0[\gamma^\mu,\gamma^\nu] \end{aligned} \right.$$

$= \psi^\dagger\gamma^0(\mathbb{1}+\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu})$

i.e. $\boxed{\bar\psi \to \bar\psi\Lambda_{1/2}^{-1}}$

$\Rightarrow$ • $\bar{\psi}\psi$ transforms like a scalar!

• $\bar{\psi}\gamma^\mu\psi$ 〃 〃 〃 vector!

$$\left[\ \bar{\psi}\gamma^\mu\psi \longrightarrow \bar{\psi}_a \underbrace{\Lambda_{1/2}^{-1}\gamma^\mu\Lambda_{1/2}}_{\Lambda^\mu{}_\nu\gamma^\nu_{ab}} \psi_b = \Lambda^\mu{}_\nu\, \bar{\psi}\gamma^\nu\psi\ \right]$$

• $\bar{\psi}S^{\mu\nu}\psi$ 〃 〃 〃 tensor!