lowest possible n in 4D : $n = 4$

$\rightsquigarrow$ we will consider 4×4 γ matrices here

Q: How to decompose a general $\Gamma = 4 \times 4$ matrix into basis elements Γ_i such that $\bar{\psi} \Gamma_i \psi$ has definite transformation properties under Lorentz transformations?

$\rightsquigarrow$ need to introduce one more

$$\boxed{\gamma^5 \equiv i \gamma^0 \gamma^1 \gamma^2 \gamma^3 = -\frac{i}{4!} \varepsilon^{\mu\nu\rho\sigma} \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma}$$

$\Rightarrow$
- $(\gamma^5)^\dagger = \gamma^5$
- $(\gamma^5)^2 = \mathbb{1}$
- $\{\gamma^5, \gamma^\mu\} = 0$

basis elements of 4×4 matrices (Γ_i)	#	LT properties ($\bar{\psi} \Gamma_i \psi$)
$\mathbb{1}$	1	scalar
γ^μ	4	vector
$\sigma^{\mu\nu} \equiv 2 S^{\mu\nu}$	6	tensor

γ^5	1	pseudo-scalar
$\gamma^\mu\gamma^5$	4	pseudo-vector / axial vector
	16 ✓	

• first determine $\Lambda_{1/2}$ for $\boxed{\text{reflections}}$, i.e. $\Lambda^\mu{}_\nu = \begin{pmatrix} 1&0&0&0\\ 0&-1&0&0\\ 0&0&-1&0\\ 0&0&0&-1 \end{pmatrix}$

$$\Rightarrow \Lambda_{1/2}^{-1}\gamma^\mu\Lambda_{1/2} = \Lambda^\mu{}_\nu\gamma^\nu = \begin{cases} \gamma^0 & (\mu=0) \\ -\gamma^i & (\mu=i) \end{cases}$$

solution : $\boxed{\Lambda_{1/2} = \eta_P\gamma^0}$ $\Rightarrow \Lambda_{1/2}^{-1} = \eta_P^*\gamma^0$

phase, i.e. $|\eta_P| = 1$

$$\Rightarrow \bullet\ \underset{(t,\vec{x})}{\bar\psi}\gamma^5\underset{(t,\vec{x})}{\psi} \xrightarrow{\vec{x}\to-\vec{x}} \bar\psi\Lambda_{1/2}^{-1}\gamma^5\Lambda_{1/2}\psi = \bar\psi\gamma^0\underbrace{\gamma^5\gamma^0}_{-\gamma^0\gamma^5}\psi = -\bar\psi\gamma^5\psi\ ✓$$

$$\bullet\ \bar\psi\gamma^\mu\gamma^5\psi \xrightarrow{\vec{x}\to-\vec{x}} \bar\psi\gamma^0\gamma^\mu\gamma^5\gamma^0\psi = \bar\psi\begin{Bmatrix} \gamma^0 & (\mu=0) \\ -\gamma^i & (\mu=i) \end{Bmatrix}\underbrace{\gamma^0\gamma^5\gamma^0}_{-\gamma^5}\psi$$

$$= -\bar\psi\gamma^\mu\gamma^5\psi\begin{cases} +1 & \text{for } \mu=0 \\ -1 & \text{for } \mu=i \end{cases}$$

✓

representations of Dirac matrices

lowest possible n in 4D : $n=4$

Pauli matrices

"Weyl" or "chiral" rep : $\boxed{\gamma^0 = \begin{pmatrix} 0_{2\times2} & \mathbb{1}_{2\times2} \\ \mathbb{1}_{2\times2} & 0_{2\times2} \end{pmatrix} ; \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}}$

$$\Rightarrow \boxed{\gamma^5 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}}$$

$\Rightarrow$ boosts : $S^{0i} = \frac{i}{4}[\gamma^0, \gamma^i] = -\frac{i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} = -(S^{0i})^+$

rotations : $S^{ij} = \frac{i}{4}[\gamma^i, \gamma^j] = \frac{1}{2}\varepsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$

$$\equiv \frac{1}{2}\varepsilon^{ijk}\Sigma^k = (S^{ij})^+$$

4. The Dirac equation

goal: find a relativistic wave equation for Dirac spinors

⤳ there exists a 1st order Lorentz-invariant equation!

$$\boxed{(i\gamma^\mu \partial_\mu - m)\,\psi(x) = 0} \quad \text{"Dirac equation"} \quad (\#)$$

$$\Rightarrow \quad 0 = (-i\gamma^\nu \partial_\nu - m)(i\gamma^\mu \partial_\mu - m)\psi$$

$$= (\gamma^\nu \gamma^\mu \partial_\nu \partial_\mu + m^2)\psi \qquad | \; \partial_\nu \partial_\mu = \partial_\mu \partial_\nu$$

$$= \Big(\underbrace{\tfrac{1}{2}\{\gamma^\nu, \gamma^\mu\}}_{g^{\mu\nu}} \partial_\nu \partial_\mu + m^2\Big)\psi$$

$$= (\partial^2 + m^2)\psi$$

$\Rightarrow$ every spinor field satisfying (#) also satisfies KG eq., i.e. correct p^μ-m relation!

$$(\Leftrightarrow) \quad \boxed{\mathcal{L} = \bar{\psi}(i\not\partial - m)\psi} \quad \text{where } \not A \equiv \gamma^\mu A_\mu$$

Weyl spinors

recall block-diagonal form of $S^{\mu\nu}$

$\rightsquigarrow$ Dirac rep. of the Lorentz group is reducible !

$$\psi \equiv \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \quad : \text{left-/righthanded "Weyl spinors"}$$

(2 components $\leftrightarrow$ 4 = of Dirac spinors)

$$\gamma^5 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}$$

$$\Rightarrow \boxed{\underbrace{\psi_L}_{=\begin{pmatrix} \psi_L \\ 0 \end{pmatrix}} = \underbrace{\left(\frac{1-\gamma^5}{2}\right)}_{\equiv P_L = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} \psi \, , \quad \underbrace{\psi_R}_{=\begin{pmatrix} 0 \\ \psi_R \end{pmatrix}} = \underbrace{\frac{1+\gamma^5}{2}}_{\equiv P_R = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}} \psi}$$

$\Rightarrow$ Dirac eq.: $0 = (i\gamma^\mu \partial_\mu - m)\psi$

$$= \begin{pmatrix} -m & i(\partial_0 + \vec{\sigma}\cdot\vec{\nabla}) \\ i(\partial_0 - \vec{\sigma}\cdot\vec{\nabla}) & -m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \qquad \left| \begin{array}{l} \sigma^\mu \equiv (1, \vec{\sigma}) \\ \tilde{\sigma}^\mu = (1, -\vec{\sigma}) \end{array} \right.$$

$$= \begin{pmatrix} -m & i\sigma\cdot\partial \\ i\tilde{\sigma}\cdot\partial & -m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

$\Rightarrow$ • $m \neq 0$ mixes ψ_R, ψ_L

• $m = 0$: $\boxed{\begin{array}{l} \tilde{\sigma} \cdot \partial \psi_L = 0 \\ \sigma \cdot \partial \psi_R = 0 \end{array}}$ "Weyl equations"

$\rightsquigarrow$ neutrinos...

Conserved currents

• "vector current" $j^\mu \equiv \bar{\psi} \gamma^\mu \psi$

$$\Rightarrow \partial_\mu j^\mu = \underbrace{(\partial_\mu \bar{\psi}) \gamma^\mu}_{-im\bar{\psi}} \psi + \bar{\psi} \underbrace{\gamma^\mu \partial_\mu \psi}_{im\psi}$$

Dirac eq:
$(i\partial\!\!\!/ + m)\psi = 0$
$\Rightarrow \psi^\dagger(-i\overleftarrow{\partial\!\!\!/}^\dagger + m) = 0$
$\Rightarrow \psi^\dagger(-i\overleftarrow{\partial\!\!\!/}^\dagger + m)\gamma^0 = 0$
$\Rightarrow \underbrace{\psi^\dagger \gamma^0}_{\bar{\psi}}(-i\overleftarrow{\partial\!\!\!/} + m) = 0$

$$\Rightarrow \boxed{\partial_\mu j^\mu = 0}$$

• "axial vector current"

$$j^{\mu 5} \equiv \bar{\psi} \gamma^\mu \gamma^5 \psi$$

$$\Rightarrow \boxed{\partial_\mu j^{\mu 5} = \ldots = 2im\bar{\psi}\gamma^5\psi}$$

$\rightsquigarrow$ conserved if $m = 0$!

similar: $j_L^\mu \equiv \bar{\psi} \gamma^\mu \left(\frac{1-\gamma^5}{2}\right) \psi = \bar{\psi}_L \gamma^\mu \psi_L$

$$j_R^\mu \equiv \bar{\psi} \gamma^\mu \left(\frac{1+\gamma^5}{2}\right) \psi = \bar{\psi}_R \gamma^\mu \psi_R$$

$\rightsquigarrow$ ψ_L and ψ_R can have different charges!