free-particle solutions

ψ obeys KG eq.

$$\Rightarrow \boxed{\psi(x) = u(p)\, e^{-ipx} + v(p)\, e^{+ipx}} \quad \text{with } p^2 = m^2,\ p^0 > 0$$

4-component spinors, independent of x

a) positive frequency : determine $u(p)$

strategy : i) use Dirac eq. for $p = p_{rest} = (m, \vec{0})$

ii) then boost with $\Lambda_{\frac{1}{2}}$ to arbitrary p^μ

i) $(i\partial\!\!\!/ - m)\psi = 0 \Rightarrow (\underbrace{p\!\!\!/}_{p_\mu \gamma^\mu} - m)\, u(p) = 0 \quad | \, p = p_{rest}$

$$\Rightarrow (m\gamma^0 - m)\, u(p_{rest}) = 0$$

$$\Leftrightarrow m \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} u(p_{rest}) = 0$$

$$\Rightarrow \boxed{u(p_{rest}) \propto \begin{pmatrix} \xi \\ \xi \end{pmatrix}}$$

ξ : arbitrary 2 - component spinor

=> two independent possibilities: ξ^r ; $r = 1,2$

with $\boxed{\xi^{r\dagger} \xi^s = \delta^{rs}}$

$(\rightsquigarrow \xi = a \cdot \xi^1 + b \xi^2)$

interpretation: recall rotation generator:

$$S^{ij} = \frac{1}{2} \varepsilon^{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$$

block diagonal => ξ transforms exactly like a 2-component spinor (spin $\frac{1}{2}$) in QM!

e.g. $\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$: spin up along z-direction

ii) now boost

- boost of a 4-vector:

$$\begin{pmatrix} E \\ \vec{p} \end{pmatrix} = p^\mu = \Lambda^\mu{}_\nu \, p^\nu_{rest} = \Lambda^\mu{}_\nu \begin{pmatrix} m \\ 0 \end{pmatrix}$$

$$\Lambda = \exp\left[-\frac{i}{2} \omega_{\rho\sigma} (\mathcal{J}^{\rho\sigma})^{\cancel{\mu}}{}_{\cancel{\nu}}\right]^\mu{}_\nu$$

exercises: $\omega_{10} = -\omega_{01} \equiv \eta$

"rapidity"

(describes boost in x-direction)

general boost:

$$\omega_{\rho\sigma} = \eta \left(\begin{array}{c|c} 0 & \hat{p}^T \\ \hline -\hat{p} & 0_{3\times3} \end{array}\right)$$

$\hat{p} \equiv \frac{\vec{p}}{|\vec{p}|}$

$$\Rightarrow \Lambda^\mu{}_\nu = \exp\left[-i\,\omega_{0i}\,(\tilde{J}^{0i})^\mu{}_\nu\right]$$

$$(\tilde{J}^{0i})_{\mu\nu} = i\left(\delta^0_\mu \delta^i_\nu - \delta^i_\mu \delta^0_\nu\right)$$

$$= i\left(\begin{array}{c|c} 0 & \hat{e}_i^T \\ \hline -\hat{e}_i & 0_{3\times3} \end{array}\right)_{\mu\nu}$$

$\hat{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ldots$

$$\Rightarrow (\tilde{J}^{0i})^\mu{}_\nu = i\begin{pmatrix} 0 & \hat{e}_i^T \\ +\hat{e}_i & 0 \end{pmatrix}^\mu{}_\nu$$

$$= \exp\left[\eta\begin{pmatrix} 0 & \hat{p}^T \\ \hat{p} & 0_{3\times3} \end{pmatrix}\right]$$

$$= \mathbb{1} + \eta\begin{pmatrix} 0 & \hat{p}^T \\ \hat{p} & 0_{3\times3} \end{pmatrix} + \frac{\eta^2}{2}\begin{pmatrix} 1 & 0 \\ 0 & \hat{p}\hat{p}^T \end{pmatrix}$$

$$+ \frac{\eta^3}{3!}\begin{pmatrix} 0 & \hat{p}^T \\ \hat{p} & 0 \end{pmatrix} + \ldots$$

$$\Rightarrow \Lambda^\mu{}_\nu \begin{pmatrix} m \\ 0 \end{pmatrix}^\nu = m\,\Lambda^\mu{}_0$$

$$\Rightarrow \boxed{\boxed{\begin{aligned} E &= m\Lambda^0{}_0 = m\cdot\cosh\eta \\ p^i &= m\Lambda^i{}_0 = m\cdot\hat{p}^i\sinh\eta \end{aligned}}}$$

- now boost spinor correspondingly:

$$u(p) = \Lambda_{1/2}\, u(p_{rest})$$

where $\Lambda_{1/2} = \exp[-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}]$ $\qquad \Big|\ S^{0i} = -\frac{i}{2}\begin{pmatrix}\sigma^i & 0 \\ 0 & -\sigma^i\end{pmatrix}$

$$= \exp[-i\,\omega_{0i}\,S^{0i}]$$

$$= \exp[-\frac{\eta}{2}\begin{pmatrix}\hat{p}\cdot\vec{\sigma} & 0 \\ 0 & -\hat{p}\cdot\vec{\sigma}\end{pmatrix}]$$

$$= \mathbb{1} - \frac{\eta}{2}\begin{pmatrix}\hat{p}\cdot\vec{\sigma} & 0 \\ 0 & -\hat{p}\cdot\vec{\sigma}\end{pmatrix} + \frac{1}{2}\left(\frac{\eta}{2}\right)^2\begin{pmatrix}(\hat{p}\cdot\vec{\sigma})^2 & 0 \\ 0 & (\hat{p}\cdot\vec{\sigma})^2\end{pmatrix}$$

$\rightarrow \hat{p}_i\hat{p}_j\underbrace{\sigma_i\sigma_j}$

$\rightarrow \frac{1}{2}\{\sigma_i,\sigma_j\}$

$= \hat{p}_i\hat{p}_j\{\delta_{ij} + \cancel{\varepsilon_{ijk}\sigma_k}\}$

$= 1$

$\Rightarrow (\quad) = \mathbb{1}$!

$$-\frac{1}{3!}\left(\frac{\eta}{2}\right)^3\begin{pmatrix}\hat{p}\vec{\sigma} & 0 \\ 0 & \hat{p}\cdot\vec{\sigma}\end{pmatrix}$$

$$\Rightarrow \Lambda_{1/2} = \cosh\frac{\eta}{2}\cdot \mathbb{1} - \sinh\frac{\eta}{2}\begin{pmatrix}\hat{p}\cdot\vec{\sigma} & 0\\ 0 & -\hat{p}\cdot\vec{\sigma}\end{pmatrix}$$

$$= \begin{pmatrix} e^{\frac{\eta}{2}}\left(\frac{1-\hat{p}\cdot\vec{\sigma}}{2}\right) + e^{-\frac{\eta}{2}}\left(\frac{1+\hat{p}\cdot\vec{\sigma}}{2}\right) & 0 \\ 0 & e^{\frac{\eta}{2}}\left(\frac{1+\hat{p}\cdot\vec{\sigma}}{2}\right) + e^{-\frac{\eta}{2}}\left(\frac{1-\hat{p}\cdot\vec{\sigma}}{2}\right)\end{pmatrix}$$

$$\left|\; e^{\pm\frac{\eta}{2}} = \cosh\frac{\eta}{2} \pm \sinh\frac{\eta}{2}\right.$$

$$= \sqrt{\cosh\eta \pm \sinh\eta}$$

$$= \sqrt{E/m \pm |\vec{p}|/m}$$

$$= \frac{1}{2\sqrt{m}}\begin{pmatrix} \sqrt{E+|\vec{p}|} + \sqrt{E-|\vec{p}|} - \hat{p}\cdot\vec{\sigma}\left(\sqrt{E+|\vec{p}|} - \sqrt{E-|\vec{p}|}\right) & 0 \\ 0 & \text{''} + \text{''} \end{pmatrix}$$

$$\left|\; \sqrt{E+|\vec{p}|} \pm \sqrt{E-|\vec{p}|} = \sqrt{(E+|\vec{p}|) + (E-|\vec{p}|) \pm 2\underbrace{\sqrt{E^2-\vec{p}^{\,2}}}_{m}}\right.$$

$$= \sqrt{2(E\pm m)}$$

$$= \frac{1}{\sqrt{2m}}\begin{pmatrix} \sqrt{E+m} - \sqrt{E-m}\;\hat{p}\cdot\vec{\sigma} & 0 \\ 0 & \sqrt{E+m} + \sqrt{E-m}\;\hat{p}\cdot\vec{\sigma}\end{pmatrix}$$

$$\left|\; |\vec{p}| = \sqrt{E^2-m^2} = \sqrt{E+m}\,\sqrt{E-m}\right.$$

$$\Rightarrow \sqrt{E-m}\,\hat{p} = \frac{\vec{p}}{\sqrt{E+m}}$$

$$= \frac{1}{\sqrt{2m(E+m)}} \begin{pmatrix} E+m-\vec{p}\cdot\vec{\sigma} & 0 \\ 0 & E+m+\vec{p}\cdot\vec{\sigma} \end{pmatrix}$$

$$\Rightarrow u(p) = \Lambda_{1/2}\, u(p_{rest})$$

r = 1,2

$$\Rightarrow \boxed{u^r = \frac{1}{\sqrt{2(E+m)}} \begin{pmatrix} [E+m-\vec{p}\cdot\vec{\sigma}]\,\xi^r \\ [E+m+\vec{p}\cdot\vec{\sigma}]\,\xi^r \end{pmatrix}}$$

normalization convention

NB: two independent solutions! ("spin up and down")

example: $\vec{p} = (0,0,p) \Rightarrow \vec{p}\cdot\vec{\sigma} = p\sigma^3 = \begin{pmatrix} p & 0 \\ 0 & -p \end{pmatrix}$

- $E+m \pm p = \sqrt{\underbrace{(E+m)^2 + p^2}_{} \pm 2p(E+m)}$

$\overset{\shortparallel}{E^2 - m^2}$

$$= \sqrt{2}\,\sqrt{E^2 + Em \pm p(E+m)}$$

$$= \sqrt{2}\,\sqrt{E+m}\,\sqrt{E \pm p}$$

$$\Rightarrow u^r(p) = \begin{pmatrix} \sqrt{E-p} & 0 & 0 & 0 \\ 0 & \sqrt{E+p} & 0 & 0 \\ 0 & 0 & \sqrt{E+p} & 0 \\ 0 & 0 & 0 & \sqrt{E-p} \end{pmatrix} \begin{pmatrix} \xi^r \\ \xi^r \end{pmatrix}$$

- normalization:

$$u^{r+} u^s = \frac{1}{2(E+m)} \left(\xi^{r+} [E+m - \vec{p}\vec{\sigma}],\ \xi^{r+} [E+m+\vec{p}\vec{\sigma}] \right) \times$$

$\sigma^{i+} = \sigma^i$

$$\times \begin{pmatrix} [E+m - \vec{p}\vec{\sigma}] \xi^s \\ [E+m+\vec{p}\vec{\sigma}] \xi^s \end{pmatrix}$$

$$= \frac{1}{2(E+m)} \xi^{r+} \underbrace{\left(\underbrace{[E+m-\vec{p}\vec{\sigma}]^2}_{2\times 2 \text{ matrix!}} + [E+m+\vec{p}\vec{\sigma}]^2 \right)}_{2(E+m)^2 \times \mathbb{1}_{2\times2} + 2(\vec{p}\vec{\sigma})^2} \xi^s$$

$p_i p_j \sigma_i \sigma_j = \vec{p}^2 \cdot \mathbb{1}_{2\times 2}$

$$= \frac{(E+m)^2 + \vec{p}^2}{E+m} \underbrace{\xi^{r+} \xi^s}_{\delta^{rs}}$$

$|\ \vec{p}^2 = (E+m)(E-m)$

$$= \underline{\underline{2E\delta^{rs}}}$$

similar: $\boxed{\bar{u}^r u^s = \ldots = 2m\delta^{rs}}$

b) negative frequency solutions

$$\psi(x) = v(p)\, e^{+ipx} \quad ; \; p^2 = m^2, \; p^0 > 0$$

$\rightsquigarrow \ldots$

$v^s(p_{rest}) \propto \begin{pmatrix} \eta^s \\ \eta^s \end{pmatrix}$

$$\boxed{v^s = \frac{1}{\sqrt{2(E+m)}} \begin{pmatrix} [E+m-\vec{p}\vec{\sigma}]\,\eta^s \\ -[E+m+\vec{p}\vec{\sigma}]\,\eta^s \end{pmatrix}}$$

with $\eta^{r\dagger}\eta^s = \delta^{rs}$ $\quad (r,s = 1,2)$

$\rightsquigarrow$ also 2 solutions, in total 4

$\Rightarrow (\ldots)$ $\boxed{\begin{aligned} v^{r\dagger} v^s &= +2E\delta^{rs} \\ \bar{v}^r v^s &= -2m\delta^{rs} \end{aligned}}$

also $(\ldots)$

$$\bar{u}^r(p)\, v^s(p) = \bar{v}^r(p)\, u^s(p) = 0$$

$$u^{r\dagger}(p)\, v^s(-p) = v^{r\dagger}(p)\, u^s(-p) = 0$$

Spin sums

$$\boxed{\begin{aligned} \sum_{s=1,2} u^s(p)\,\bar{u}^s(p) &= \not{p} + m \\ \sum_{s=1,2} v^s(p)\,\bar{v}^s(p) &= \not{p} - m \end{aligned}}$$

$$\sum_{s=1,2} u^s(p)\, u^{s\dagger}(p)\, \gamma^0$$

$$= \frac{1}{2(E+m)} \sum_{s=1,2} \begin{pmatrix} [E+m-\vec{p}\vec{\sigma}]\,\xi^s \\ [E+m+\vec{p}\vec{\sigma}]\,\xi^s \end{pmatrix} \times \left(\xi^{s\dagger}[E+m-\vec{p}\vec{\sigma}],\ \xi^{s\dagger}[E+m+\vec{p}\vec{\sigma}] \right)$$

$$\times \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

$$= \frac{1}{2(E+m)} \sum_{s=1,2} \begin{pmatrix} [E+m-\vec{p}\vec{\sigma}]\,\xi^s \\ [E+m+\vec{p}\vec{\sigma}]\,\xi^s \end{pmatrix} \times \left(\xi^{s\dagger}[E+m+\vec{p}\vec{\sigma}],\ \xi^{s\dagger}[E+m-\vec{p}\vec{\sigma}] \right)$$

$$\Big|\ \sum_{s=1,2} \xi^s \xi^{s\dagger} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \frac{1}{2(E+m)} \begin{pmatrix} (E+m)^2 - \underbrace{(\vec{p}\vec{\sigma})^2}_{\vec{p}^{\,2}} & (E+m)^2 - 2\vec{p}\vec{\sigma}(E+m) + \vec{p}^{\,2} \\ (E+m)^2 + 2\vec{p}\vec{\sigma}(E+m) + \vec{p}^{\,2} & (E+m)^2 - \vec{p}^{\,2} \end{pmatrix}$$

NB: every entry is a 2x2 matrix!

$$\left|\ \vec{p}^{\,2} = (E+m)(E-m)\right.$$

$$= \frac{1}{2}\begin{pmatrix} E+m-(E-m) & E+m+(E-m)-2\vec{p}\vec{\sigma} \\ E+m+(E-m)+2\vec{p}\vec{\sigma} & E+m-(E-m) \end{pmatrix}$$

$$= \begin{pmatrix} m & E-\vec{p}\sigma \\ E+\vec{p}\sigma & m \end{pmatrix}$$

$$= m\cdot \mathbb{1}_{4\times4} + p_\mu \gamma^\mu \quad \square \quad \lrcorner$$