5. Quantizing the Dirac field

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi = \psi^+(i\partial_t + i\gamma^0\vec{\gamma}\vec{\nabla} - m)\psi$$

$$\Rightarrow \pi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = i\psi^+$$

$$\Rightarrow H = \int d^3x\,(\pi\dot{\psi} - \mathcal{L})$$

$$= \int d^3x\, \psi^+ \underbrace{(-i\gamma^0\vec{\gamma}\cdot\vec{\nabla} + m\gamma^0)}_{\equiv h_D}\psi$$

energy eigenvalues

goal: diagonalize H like for scalar field

$\to$ need to identify all 4 eigenfunctions of $(\gamma)h_D$!

recall Dirac equation:

$$[i\cancel{\gamma^0}\partial_t + \underbrace{i\vec{\gamma}\vec{\nabla} - m}_{-\cancel{\gamma_0}h_D}]u^s(\vec{p})e^{-ipx} = 0 \qquad \Big|\, i\partial_t e^{-ipx} = p^0 e^{-ip^0t + i\vec{p}\vec{x}}$$

$$\Rightarrow h_D u^s(\vec{p})e^{+i\vec{p}\vec{x}} = +p^0 u^s(\vec{p})e^{+i\vec{p}\vec{x}}$$

$$\bullet\; h_D\, v^s(\underset{-\vec p}{\vec p})\, e^{\underset{+}{-} i\vec p\vec x} = -p^0\, v^s(\underset{-\vec p}{\vec p})\, e^{\underset{+}{-} i\vec p\vec x}$$

expand in this basis, promote ψ to operator:

$$\boxed{\psi_a(\vec x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec p}}}\, e^{+i\vec p\vec x} \sum_{s=1,2}\left(a^s_{\vec p}\, u^s_a(\vec p) + \tilde b^s_{-\vec p}\, v^s_a(-\vec p)\right)}$$

Schrödinger picture
$\rightsquigarrow$ no t-dependence

$$\Rightarrow H = \int d^3x\, \psi^\dagger h_D \psi$$

$$= \int d^3x \int \frac{d^3p'\, d^3p}{(2\pi)^6}\frac{1}{2\sqrt{E_p E_{p'}}}\, e^{-i\vec p\vec x}\sum_{r,s=1,2}\left(a^{s\dagger}_{\vec p}\, u^{s\dagger}(\vec p) + \tilde b^{s\dagger}_{-\vec p}\, v^{s\dagger}(-\vec p)\right)$$

$$\times\, \underbrace{h_D\, e^{+i\vec p'\vec x}\left(a^r_{\vec p'}\, u^r(\vec p') + \tilde b^r_{-\vec p'}\, v^r(-\vec p')\right)}_{E_{p'}\, e^{+i\vec p'\vec x}\left(a^r_{\vec p'}\, u^r(\vec p') - \tilde b^r_{-\vec p'}\, v^r(-\vec p')\right)}$$

$$\Big|\; \int d^3x\, e^{i\vec x(\vec p'-\vec p)} = (2\pi)^3\delta^{(3)}(\vec p-\vec p')$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\cancel{E_p}}\sum_{r,s}\left(a^{s\dagger}_{\vec p}\, u^{s\dagger}(\vec p) + \tilde b^{s\dagger}_{-\vec p}\, v^{s\dagger}(-\vec p)\right)$$

$$\times \left(a^r_{\vec p}\, u^r(\vec p) - \tilde b^r_{-\vec p}\, v^r(-\vec p)\right)$$

$\cdot\; u^{s\dagger} u^{r} = 2E_p \delta^{rs}$ $\qquad$ $\bullet\; u^{s\dagger}(\vec{p}) v^{r}(-\vec{p})$

$\cdot\; v^{s\dagger} v^{r} = 2E_p \delta^{rs}$ $\qquad$ $= v^{s\dagger}(-\vec{p}) u^{r}(\vec{p}) = 0$

$$\boxed{H = \int \frac{d^3p}{(2\pi)^3} E_p \sum_s \left(a^{s\dagger}_{\vec{p}} a^{s}_{\vec{p}} - \tilde{b}^{s\dagger}_{\cancel{-}\vec{p}} \tilde{b}^{s}_{\cancel{-}\vec{p}} \right)}$$

problem...

getting rid of negative energies

postulate anti- commutation relations!

$$\boxed{\{a^s_{\vec{p}}, a^{r\dagger}_{\vec{p}'}\} \overset{!}{=} \{\tilde{b}^s_{\vec{p}}, \tilde{b}^{r\dagger}_{\vec{p}'}\} \overset{!}{=} (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{p}')\delta^{rs}}$$

$+ \{\,,\,\} = 0$ otherwise

$\Rightarrow b^s_{\vec{p}} \equiv \tilde{b}^{s\dagger}_{\vec{p}}$ satisfies the same relations!

$$\Rightarrow a) \quad \boxed{H = \int \frac{d^3p}{(2\pi)^3} E_p \sum_{s=1,2} \left(a^{s\dagger}_{\vec{p}} a^{s}_{\vec{p}} + b^{s\dagger}_{\vec{p}} b^{s}_{\vec{p}} \right)} \text{ "}-\infty\text{"}$$

↳ const, disregard like in scalar case!

same: $\vec{P}$ = $\vec{p}$ = " "

b) $\{\psi(\vec{x}), i\psi^\dagger(\vec{y})\}$

$$= i\int \frac{d^3p\, d^3p'}{(2\pi)^6} \frac{1}{2\sqrt{E_p E_{p'}}} \sum_{r,s} e^{i\vec{p}\vec{x} - i\vec{p}'\vec{y}} \left\{ \left(a^s_{\vec{p}} u^s(\vec{p}) + b^{s\dagger}_{-\vec{p}} v^s(-\vec{p})\right), \left(a^{r\dagger}_{\vec{p}'} u^{r\dagger}(\vec{p}') + b^r_{-\vec{p}'} v^{r\dagger}(-\vec{p}')\right) \right\}$$

$$\{a^s_{\vec{p}}, a^{r\dagger}_{\vec{p}'}\} u^s(\vec{p}) u^{r\dagger}(\vec{p}') + \{b^{s\dagger}_{-\vec{p}}, b^r_{-\vec{p}'}\} v^s(-\vec{p}) v^{r\dagger}(-\vec{p}')$$

$$= i\int \frac{d^3p}{(2\pi)^3} \frac{e^{i\vec{p}(\vec{x}-\vec{y})}}{2E_p} \underbrace{\sum_s \left(u^s(\vec{p}) \bar{u}^{s}(\vec{p}') + v^s(-\vec{p}) \bar{v}^{s}(-\vec{p}) \right)}_{\cancel{\not p} + \cancel{m} + (\gamma^0 p^0 - \cancel{\gamma^i p^i} - \cancel{m})} \gamma^0$$

$$\cancel{\not p} = \gamma^0 p^0 + \cancel{\gamma^i p^i}$$

$$= i\int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}(\vec{x}-\vec{y})} \frac{1}{\cancel{E_p}} \cancel{p^0} \times \mathbb{1}_{4\times4}$$

$$\Leftrightarrow \boxed{\{\psi_a(\vec{x}), i\psi^\dagger_b(\vec{y})\} = i\,\delta(\vec{x}-\vec{y})\,\delta_{ab}}$$

$\rightsquigarrow$ "expected" (but $[\,,\,] \to \{\,,\,\}$)

Spin → statistics

$$\{a^{s\dagger}_{\vec p}, a^{r\dagger}_{\vec r}\} = 0 \Rightarrow \bullet\ (a^{s\dagger}_{\vec p})^2 |0\rangle = 0 \quad (\ast)$$

⇝ only one particle in state $(\vec p, s)$ possible!

$$\bullet\ a^{s\dagger}_{\vec p}\, a^{r\dagger}_{\vec r} |0\rangle = - a^{r\dagger}_{\vec r}\, a^{s\dagger}_{\vec p} |0\rangle$$

⇒ particles described by Dirac equation
(+ anti-commutation relations)
obey Fermi-Dirac statistics!

more general theorem by Pauli:

1) Lorentz invariance
2) $E_{\vec p} > 0$
3) positive norms
4) causality

⇒ particles with integer / half-integer spin obey Bose-Einstein / Fermi-Dirac statistics!

remark: for every $(\vec{p}, s)$, there exists only two states defined by $|0\rangle$ and $|1\rangle$

$\Rightarrow$ 2 options: i) $b|0\rangle \equiv 0 \rightsquigarrow b^{+}|0\rangle \equiv |1\rangle$

ii) $\tilde{b}|0\rangle \equiv 0, \ \tilde{b}^{+}|0\rangle = |1\rangle$

$(\Leftrightarrow \ b|0\rangle = |1\rangle, \ b^{+}|0\rangle = 0)$

physical choice: denote the state of lower energy ("vacuum") with $|0\rangle$!

$\Rightarrow$ i) $\langle 0|H = E b^{+} b|0\rangle = 0$

$< \langle 1|E b^{+} b|1\rangle$

ii) $\langle 0|H = -E\hat{b}^{+}\hat{b}|0\rangle = 0 > \langle 1|-E\hat{b}^{+}\hat{b}|1\rangle$

$= -E$ $\lightning$

- full $x=(t,\vec{x})$ dependence

as before: Schrödinger → Heisenberg

i.e. $\psi(x) = e^{iHt}\,\psi(\vec{x})\,e^{-iHt}$

$$\Rightarrow \boxed{\psi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_{s=1,2}\left(a^s_{\vec{p}}\,u^s(p)e^{-ipx} + b^{s\dagger}_{\vec{p}}\,v^s(p)e^{+ipx}\right)}$$

$| \; p^0 = E_p = \sqrt{\vec{p}^2+m^2}$

$a^\dagger$ creates "fermions"
$b^\dagger$ creates "anti-fermions"
} both with $E_p > 0$

1-particle states normalized as before:

$$|\vec{p},s\rangle \equiv \sqrt{2E_{\vec{p}}}\;a^{s\dagger}_{\vec{p}}\,|0\rangle$$

$$\Rightarrow \langle \vec{p},r\,|\,\vec{q},s\rangle = 2E_{\vec{p}}\,(2\pi)^3\,\delta^{(3)}(\vec{p}-\vec{p}\,')\,\delta^{rs}$$

1 antifermion @ x : $\psi(x)\,|0\rangle$

1 fermion @ x : $\bar{\psi}(x)\,|0\rangle$

(electric) charge

recall: $j^\mu = \bar{\psi}\gamma^\mu\psi$ is conserved

$$\Rightarrow Q = \int d^3x\, j^0 = \int d^3x\, \psi^\dagger \psi$$

$$\vdots$$

$$= \int \frac{d^3p}{(2\pi)^3} \sum_s \left(a^{s\dagger}_{\vec p} a^s_{\vec p} + b^s_{\vec p} b^{s\dagger}_{\vec p} \right)$$

$$= \int \frac{d^3p}{(2\pi)^3} \sum_s \left(a^{s\dagger}_{\vec p} a^s_{\vec p} - b^{s\dagger}_{\vec p} b^s_{\vec p} \right) \quad [+\infty]$$

"charge of vacuum"

$\Rightarrow \begin{pmatrix} a^\dagger \\ b^\dagger \end{pmatrix}$ creates $\begin{pmatrix} \text{fermions} \\ \text{anti-fermions} \end{pmatrix}$ with charge $\begin{pmatrix} +1 \\ -1 \end{pmatrix}$

$=$ const. $\times$ electric charge