Dirac propagator

amplitude for fermion to propagate from y to x:

$$\langle 0 | \psi_a(x) \bar{\psi}_b(y) | 0 \rangle$$

$$= \int \frac{d^3p\, d^3p'}{(2\pi)^6} \frac{1}{2\sqrt{E_p E_{p'}}} \sum_{r,s} \langle 0 | \left(a^r_{\vec{p}} u^r_a(p) e^{-ipx} + \cancel{b^{r\dagger}_{\vec{p}} v^r_a e^{ipx}} \right)$$

$$\times \left(a^{s\dagger}_{\vec{p}'} \bar{u}^s_b(\vec{p}') e^{+ip'y} + \cancel{b^s_{\vec{p}'} \bar{v}^s_b(\vec{p}') e^{-ip'y}} \right) | 0 \rangle$$

$$\rightarrow (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{p}')\, \delta^{rs}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \underbrace{\sum_r u^r_a(\vec{p}) \bar{u}^r_b(\vec{p})}_{(\not{p}+m)_{ab}} e^{-ip(x-y)}$$

$$= (i\not{\partial}_x + m)_{ab} \underbrace{\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip(x-y)}}_{= D(x-y)\ \left[\equiv \langle 0 | \phi(x) \phi(y) | 0 \rangle \right]}$$

similar : $\langle 0 | \bar{\psi}_b(y) \psi_a(x) | 0 \rangle = \ldots = -(i\not{\partial}_x + m)_{ab} D(y-x)$

Green's functions of Dirac equation :

$$(i\not{\partial}_x - m)\, G(x-y) = i\,\delta^{(4)}(x-y)\cdot \mathbb{1}$$

$$\Leftrightarrow \int \frac{d^4p}{(2\pi)^4} (i\underbrace{\not{\partial}_x}_{\to \not{p}} - m)\, G(p)\, e^{-ip(x-y)} = \int \frac{d^4p}{(2\pi)^4}\, i\, e^{-ip(x-y)}$$

$$\Rightarrow \quad G(p) = \frac{i}{\not{p} - m} = \frac{i(\not{p}+m)}{p^2 - m^2}$$

$\not{A}\not{A} = A^2 \cdot \mathbb{1}_{4\times 4}$

$\Rightarrow$ Feynman propagator

$$\boxed{S_F(x-y)_{ab} \equiv \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{p}+m)_{ab}}{p^2 - m^2 + i\epsilon}\, e^{-ip(x-y)}}$$

$$= (i\not{\partial}_x + m)_{ab} \underbrace{\int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon}\, e^{-ip(x-y)}}_{D_F(x-y) \equiv \langle 0 | T \phi(x) \phi(y) | 0 \rangle}$$

$$= \begin{cases} \langle 0 | \psi_a(x) \bar{\psi}_b(y) | 0 \rangle & \text{for } x^0 > y^0 \\ - \langle 0 | \bar{\psi}_b(y) \psi_a(x) | 0 \rangle & \text{for } y^0 > x^0 \end{cases}$$

$$\boxed{S_F(x-y)_{ab} \equiv \langle 0| T\, \psi_a(x)\, \bar{\psi}_b(x) |0\rangle}$$

NB: Definition includes extra minus sign for every field (anti-) commutation necessary to achieve time ordering!

Spin of Dirac fermions

-> use Noether's theorem to derive angular momentum

= conserved charge from invariance under rotations

$$\psi(x) \to \psi'(x) = \Lambda_{1/2}\, \psi(\Lambda^{-1}x) \equiv \psi(x) + \theta \Delta\psi + \mathcal{O}(\theta^2)$$

small rotation by angle θ, around z-axis:

- $\Lambda_{1/2} \simeq \mathbb{1} - \frac{i}{2} \omega_{\mu\nu} S^{\mu\nu}$ $\quad | \; \omega_{12} = -\omega_{21} = \theta$

$S^{ij} = \frac{1}{2} \varepsilon^{ijk} \Sigma^k$

$\Sigma^k = \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$

$$= \mathbb{1} - \frac{i}{2} \theta\, \Sigma^3$$

- $\Lambda^{-1}x = (t, \cos\theta\, x + \sin\theta\, y, \cos\theta\, y - \sin\theta\, x, z)$

$$\simeq (t, x + \theta y, y - \theta x, z) + \mathcal{O}(\theta^2)$$

$$\Rightarrow \theta \Delta\psi = \Lambda_{\frac{1}{2}} \psi(\Lambda^{-1}x) - \psi(x)$$

$$= \left(\mathbb{1} - \frac{i}{2}\theta\Sigma^3\right)\psi(t, x+\theta y, y-\theta x, z) - \psi(x)$$

$$= \theta\left(y\partial_x - x\partial_y - \frac{i}{2}\Sigma^3\right)\psi(x)$$

$\Rightarrow$ conserved charged density:

$$j^0 = \frac{\partial\mathcal{L}}{\partial(\partial_0\psi)}\Delta\psi = -i\psi^\dagger\left(x\partial_y - y\partial_x + \frac{i}{2}\Sigma^3\right)\psi$$

$\rightsquigarrow$ analogously for rotations around x, y axis

$$\boxed{\boxed{\vec{J} = \int d^3x\, \psi^\dagger\left\{\vec{x}\times(-i\vec{\nabla}) + \frac{1}{2}\vec{\Sigma}\right\}\psi}}$$

now consider particles at rest: $a^{s\dagger}_{\vec{p}=0}|0\rangle$ [and $b^{s\dagger}_{\vec{p}=0}|0\rangle$]

$J_z|0\rangle = 0$

$$\to J_z a^{s\dagger}_0|0\rangle = [J_z, a^{s\dagger}_0]|0\rangle$$

$$\vdots$$

$$= \frac{1}{2m}\sum_r u^{r\dagger}(\vec{0})\frac{\Sigma^3}{2}u^s(\vec{0})a^{r\dagger}_0|0\rangle \quad \Big| \quad u(0) = \sqrt{m}\begin{pmatrix}\xi\\ \xi\end{pmatrix}$$

$$v(0) = \sqrt{m}\begin{pmatrix}\eta\\ -\eta\end{pmatrix}$$

$$= \sum_r \xi^{r\dagger} \frac{\sigma^3}{2} \xi^s a_0^{r\dagger} |0\rangle$$ | choose $\xi = \begin{pmatrix}1\\0\end{pmatrix}$ or $\begin{pmatrix}0\\1\end{pmatrix}$

(eigenstates of σ^3)

$$\Rightarrow \boxed{\begin{aligned} J_z a_0^{s\dagger}|0\rangle &= \pm\frac{1}{2} a_0^{s\dagger}|0\rangle \\ J_z b_0^{s\dagger}|0\rangle &= \mp\frac{1}{2} b_0^{s\dagger}|0\rangle \end{aligned}}$$

upper sign $\xi/\eta = \begin{pmatrix}1\\0\end{pmatrix}$

lower sign $\xi/\eta = \begin{pmatrix}0\\1\end{pmatrix}$

⌜similar (formal) argument for Klein Jordan particles: "$\mathbb{1}_{1/2} = \mathbb{1}$" (scalar) $\to$ no term like $\vec{\Sigma}$

$\to \dots \to J_z a_0^\dagger |0\rangle = 0 \Leftrightarrow$ spin $= 0$ ⌟

Discrete symmetries C, P, T

- parity P

classical : $\psi \xrightarrow{\vec{x} \to -\vec{x}} \Lambda_{1/2}\, \psi$ $\quad \Lambda_{1/2}(P) = \eta\, \gamma^0$

QM : $\psi |0\rangle \longrightarrow P\psi|0\rangle = \underbrace{P\psi P^{-1}}_{\psi' = P\psi P} P|0\rangle$

want : $P a^s_{\vec{p}} P = \eta_a\, a^s_{-\vec{p}} \qquad P b^s_{\vec{p}} P = \eta_b\, b^s_{-\vec{p}}$

phases : observables always contain an even number of operators, and should remain unchanged after applying P twice

$\Rightarrow \eta_a^2 = \pm 1 \; ; \; \eta_b^2 = \pm 1$

(see P&S)

$$\underbrace{\boxed{P\psi(t,\vec{x})P = \eta_a \gamma^0 \psi(t,-\vec{x})}}_{\bar{\psi} \;\to\; \eta_a^* \bar{\psi}(t,\vec{x})\gamma^0} \quad \text{and} \quad \boxed{\eta_a \cdot \eta_b = -1}$$

$\rightsquigarrow$ can set $\eta_a = -\eta_b [= 1]$

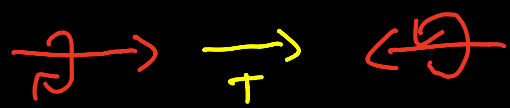
$\Rightarrow P a^{s\dagger}_{\vec{p}} b^{r\dagger}_{\vec{q}} |0\rangle = \ominus\, a^{s\dagger}_{-\vec{p}} b^{r\dagger}_{-\vec{q}} |0\rangle$

time reversal T

want: $a_{\vec p} \longrightarrow a_{-\vec p}$

$\wedge\ \psi(t,\vec x) \rightarrow \psi(-t,\vec x)$

$\wedge$ spin flip!



$\Rightarrow T a^{s}_{\vec p} T = a^{-s}_{-\vec p}$ ← flipped spin

$\boxed{T i T = -i}$ T is "anti-unitary"

(P&S)

$$\boxed{T\psi(t,\vec x)T = \gamma^1\gamma^3\psi(-t,\vec x)}$$

Charge conjugation C

(fermion $\leftrightarrow$ antifermion w/ same spin, momentum)

$C a^{s}_{\vec p} C = b^{s}_{\vec p}$

$C b^{s}_{\vec p} C = a^{s}_{\vec p}$

$\rightsquigarrow$ $\boxed{C\psi(x)C = -i\gamma^2\psi^*(x)}$

C P T

explicit representations of C, P, T allow to work out transformation of $\bar{\psi}\Gamma\psi$:

	$\bar{\psi}\psi$	$i\bar{\psi}\gamma^5\psi$	$\bar{\psi}\gamma^\mu\psi$	$\bar{\psi}\gamma^\mu\gamma^5\psi$	$\bar{\psi}\sigma^{\mu\nu}\psi$	$i\partial_\mu$
P	+	−	$(-1)^\mu = \begin{cases} + & \text{for } \mu=0 \\ - & \mu=i \end{cases}$	$-(-)^\mu$	$(-)^\mu(-)^\nu$	$(-)^\mu$
T	+	−	$(-)^\mu$	$(-)^\mu$	$-(-)^\mu(-)^\nu$	$+(-)^\mu$
C	+	+	−	+	−	$+ \to -$
CPT	+	+	−	−	+	−

Dirac Lagrangian: $\mathcal{L}_{Dirac} = \bar{\psi}(i\not{\partial} - m)\psi$

C, P, T ✓

<u>CPT theorem</u> : Any QFT that satisfies the
(Pauli) following is invariant under CPT:

- Lorentz invariance
- causality
- locality
- Hamiltonian bounded from below ("$\hat{E}_p > 0$")

even stronger : $\cancel{\text{CPT}}$ => ~~Lorentz invariance~~