6. Perturbation theory

$$\mathcal{L}_0 \longrightarrow \mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}$$

free fields: quadratic
$\Rightarrow$ linear equations of motion

non-linear terms!
$\Rightarrow$ couple different Fourier modes

$$\Rightarrow H_{int}[\phi] = \int d^3x\, \mathcal{H}_{int}[\phi, \partial\phi] = -\int d^3x\, \mathcal{L}_{int}[\phi, \partial\phi]$$

NB: • these are <u>local</u> interactions
[e.g. $\phi^2(x)\phi(y)$ not allowed]

• $\partial\phi$ dependence changes def. of π!

$$\Rightarrow \boxed{\partial_\mu \frac{\partial \mathcal{L}_0}{\partial(\partial_\mu \phi)} - \frac{\partial \mathcal{L}_0}{\partial \phi} = \underbrace{\frac{\partial \mathcal{L}_{int}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}_{int}}{\partial(\partial_\mu \phi)}}_{\text{inhomogeneous term}}} \quad (\ast)$$

$\rightsquigarrow$ Green's functions!

simplest example : "ϕ^4-theory" (~ Higgs !)

$$\mathcal{L} = \underbrace{\frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2}_{\mathcal{L}_0} \underbrace{- \frac{\lambda}{4!}\phi^4}_{\mathcal{L}_{int}}$$

$$(*) \Rightarrow \quad (\partial^2 + m^2)\phi = -\frac{\lambda}{3!}\phi^3$$

perturbation theory : $\lambda \ll 1$

$$\Rightarrow \phi = \phi_0 + \lambda \phi_1 + \lambda^2 \phi_2 + \dots$$

where $(\partial^2 + m^2)\phi_0 = 0$

$$(\partial^2 + m^2)(\cancel{\phi_0} + \cancel{\lambda}\phi_1 + \dots) = -\frac{\cancel{\lambda}}{3!}(\phi_0 + \dots)$$

$$\vdots$$

correlation functions

⇝ fundamental "building blocks" to describe (not only) interactions!

simplest example: 2-point function = Green's function:

$$\underbrace{\langle \Omega | T\phi(x)\phi(y)|\Omega\rangle}_{\substack{\text{ground state}\\ \text{of interacting theory}}} = \underbrace{\langle 0|T\phi(x)\phi(y)|0\rangle_{\text{free}}}_{=D_F(x-y)} + \mathcal{O}(\lambda)$$

goal: compute expansion in λ!

step 1: express $\phi(x)$ in terms of free-field solutions

1a) transform to interaction picture:

$$\phi(t,\vec{x}) = e^{iH(t-t_0)}\phi(t_0,\vec{x})\,e^{-iH(t-t_0)}$$

$$= \underbrace{e^{iH(t-t_0)}e^{-iH_0(t-t_0)}}_{=U^\dagger(t,t_0)}\underbrace{e^{+iH_0(t-t_0)}\phi(t_0,\vec{x})\,e^{-iH_0(t-t_0)}}_{\equiv\phi_I(t,\vec{x})}\underbrace{e^{+iH_0(t-t_0)}e^{-iH(t-t_0)}}_{\equiv U(t,t_0)}$$

ϕ_I: "interaction picture field"

$U(t,t_0)$: (#1)

$$= \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(a_{\vec{p}}e^{-ipx}+a_{\vec{p}}^{\dagger}e^{ipx}\right)\Big|_{x^0=t-t_0}$$

[possible because H_0 can be diagonalized as before]

(*) $U = (t,t_0)$: "time evolution operator"/ "interaction picture propagator"

1b) determine U in terms of ϕ_I

$$\partial_t U(t,t_0) = iH_0 e^{iH_0(t-t_0)} e^{-iH(t-t_0)} + e^{iH_0(t-t_0)}(-iH)e^{-iH(t-t_0)}$$

$$= -i\, e^{iH_0(t-t_0)} \underbrace{(H-H_0)}_{=H_{int}} e^{-iH(t-t_0)}$$

$$= -i\, \underbrace{e^{iH_0(t-t_0)} H_{int}\, e^{-iH_0(t-t_0)}}_{\equiv H_I = H_{int}[\phi_I] \left(= \frac{\lambda}{4!}\int d^3x\, \phi_I^4\right)} \cdot U(t,t_0)$$

[because $e^{iH_0}\phi^n e^{-iH_0} = \underbrace{e^{iH_0}\phi e^{-iH_0}}_{\phi_I} e^{iH_0} \dots \underbrace{e^{iH_0}\phi e^{-iH_0}}_{\phi_I} = \phi_I^n$]

$\Rightarrow$ solution : $U(t,t_0) \text{"="} \cancel{e^{-i\int_{t_0}^{t} H_I\, dt}}$

not true because $[H_I, U] \neq 0$!

$$= 1 + (-i)\int_{t_0}^{t} dt_1\, H_I(t_1) + (-i)^2 \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, \underbrace{H_I(t_1) H(t_2)}_{\text{do not}}$$

(in general)
commute!

$$+(-i)^3\int_{t_0}^{t}dt_1\int_{t_0}^{t_1}dt_2\int_{t_0}^{t_2}dt_3\, H_I(t_1)H_I(t_2)H_I(t_3)$$

$$+ \dots$$

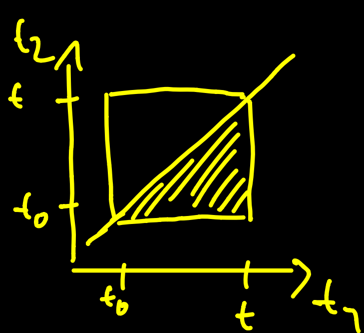
⌈apply ∂_t: each term $= -i H_I \times$ previous term ✓⌋

now simplify, noting that all terms are time-ordered:

$$\Rightarrow \int_{t_0}^{t}dt_1\int_{t_0}^{t_1}dt_2\cdots\int_{t_0}^{t_{n-1}}dt_n\, H_I(t_1)\dots H_I(t_n)$$

$$= \underbrace{\int_{t_0}^{t}dt_1\int_{t_0}^{t_1}dt_2\cdots\int_{t_0}^{t_{n-1}}dt_n}_{}\, T\{H_I(t_1)\dots H_I(t_n)\}$$

⌈



$$\Rightarrow \underbrace{\int_{t_0}^{t}dt_1\int_{t_0}^{t_1}dt_2}_{}\, f(t_1,t_2) = \frac{1}{2}\underbrace{\int_{t_0}^{t}dt_1\int_{t_0}^{t}dt_2}_{}\, f(t_1,t_2)$$

if $f(t_1,t_2) = f(t_2,t_1)$ ⌋

$$= \frac{1}{n!}\int_{t_0}^{t}dt_1\dots\int_{t_0}^{t}dt_n\, T\{H_I(t_1)\dots H_I(t_n)\}$$

$$\Rightarrow \boxed{U(t,t_0) = T\left\{\exp\left[-i\int_{t_0}^{t} dt'\, H_I(t')\right]\right\}} \quad \text{for any } t \ge t_0$$

$\Rightarrow$ (...)

- $U^\dagger = U^{-1}$
- $U^{-1}(t_1,t_2) = U(t_2,t_1)$
- $U(t_1,t_2)\,U(t_2,t_3) = U(t_1,t_3)$ for $t_1 \ge t_2 \ge t_3$

<u>step 2</u>: express $|\Omega\rangle$ in terms of free-field quantities

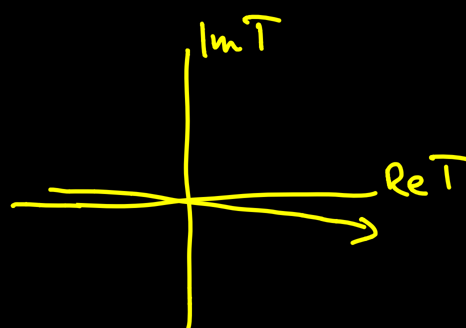
consider $$e^{-iHT}|0\rangle = \sum_n e^{-iE_nT}|n\rangle\langle n|0\rangle$$

eigenvalues and -states of H

$$(*) = e^{-iE_0T}|\Omega\rangle\langle\Omega|0\rangle + \sum_{n\neq 0} e^{-iE_nT}|n\rangle\langle n|0\rangle$$

- $\langle\Omega|H|\Omega\rangle < E_n \;\; \forall n \neq 0$!
- $\langle\Omega|0\rangle \neq 0$ (by <u>assumption</u> of small perturbation!)

now take limit $T \to \infty(1-i\varepsilon)$



(#)

$$\Rightarrow |\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \left(e^{-iE_0T}\langle\Omega|0\rangle\right)^{-1} e^{-iHT}\underbrace{|0\rangle}_{=e^{-iH_0(-T-t_0)}|0\rangle} \quad \Big| \ T \to T+t_0$$

$$= \lim_{T\to\infty(1-i\varepsilon)} \left(e^{-iE(T+t_0)}\langle\Omega|0\rangle\right)^{-1} \underbrace{e^{iH(-T-t_0)} e^{-iH_0(-T-t_0)}}_{=U^{-1}(-T,t_0)=U(t_0,-T)}|0\rangle$$

similar: $\langle\Omega| = \lim_{T\to\infty(1-i\varepsilon)} \langle 0|U(T,t_0)\left(e^{-iE_0(T-t_0)}\langle 0|\Omega\rangle\right)^{-1}$

$$\Rightarrow \langle\Omega|T\{\phi(x)\ldots\phi(y)\}|\Omega\rangle$$

$$= \lim_{T\to\infty(1-i\varepsilon)} \left(|\langle 0|\Omega\rangle|^2 e^{-iE_0(2T)}\right)^{-1}$$

$$\times \underbrace{\langle 0|U(T,t_0)\,T\{\phi(x)\ldots\phi(y)\}\,U(t_0,-T)|0\rangle}_{\langle 0|T\{\underbrace{U(T,t_0)U(t_0,x^0)}_{=U(T,x^0)}\phi_I(x)U(x^0,t_0)\ldots\underbrace{U(t_0,y^0)}_{U(x^0,y^0)}\phi_I(y)\underbrace{U(y^0,t_0)U(t_0,-T)}_{U(y^0,-T)}\}|0\rangle} \quad \Big| \ \phi(x) = U(t_0,x^0)\,\phi_I(x)\,U(x^0,t_0)$$

$$= \langle 0|T\{\phi_I(x)\ldots\phi_I(y)\,U(T,-T)\}|0\rangle$$

$$\Rightarrow \boxed{\langle \Omega | T\{\phi(x)\ldots\phi(y)\}|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|T\{\phi_I(x)\ldots\phi_I(y)\exp[-i\int_{-T}^{T} dt\, H_I(t)]\}|0\rangle}{\langle 0|T\{\exp[-i\int_{-T}^{T} dt\, H_I(t)]\}|0\rangle}}$$

NB : exact expression, but suitable for expansion in small couplings ($H_I \propto \lambda$)

$H_I \sim \phi^n$

$\Rightarrow$ need only ever to evaluate

$$\langle 0 | T\{\phi_I(x_1)\phi_I(x_2)\ldots\phi(x_m)\}|0\rangle \;!$$