Wick's theorem

goal: simplify calculations of $\langle 0 | T\{ \dots \} | 0 \rangle$

NB: drop index "I" in the following, i.e. $\phi_I(x) \to \phi(x)$
(we are always in the interaction picture!)

$$\phi(x) = \underbrace{\int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} a_p e^{-ipx}}_{\equiv \phi^+(x)} + \underbrace{\int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} a_p^\dagger e^{+ipx}}_{\phi^-(x)}$$

Def. normal order $N(\mathcal{O})$ of an operator $\mathcal{O}$:
place all $a^\dagger / \phi^-$ to the left
a / ϕ^+ to the right

$$\Rightarrow \langle 0 | N(\mathcal{O}) | 0 \rangle = 0$$

[sometimes ":$\mathcal{O}$:" is also used]

Def. contraction

$$\overline{\phi(x)\,\phi(y)} = \begin{cases} [\phi^+(x), \phi^-(y)] & \text{for } x^0 > y^0 \\ [\phi^+(y), \phi^-(x)] & \text{for } y^0 > x^0 \end{cases}$$

$$= D_F(x-y)$$

Wick's theorem

$$\boxed{T\{\phi(x_1)\dots\phi(x_n)\} = N\{\phi(x_1)\dots\phi(x_n) + \text{all possible contractions}\}}$$

$$\Rightarrow \boxed{\begin{aligned}&\langle 0|T\{\phi(x_1)\dots\phi(x_n)\}|0\rangle \\ &= \sum \text{all } \underline{\text{full}} \text{ contractions}\end{aligned}}$$

⌈ proof by induction: a) for $n=2$

b) show $n-1 \Rightarrow n$ (P&S)

a)
$$T\{\underbrace{\phi(x_1)}_{\equiv\phi_1}\underbrace{\phi(x_2)}_{\equiv\phi_2}\} = T\{\phi_1^+\phi_2^+\} + T\{\phi_1^-\phi_2^-\} + \underbrace{T\{\phi_1^+\phi_2^-\}}_{\substack{=N\{\phi_1^+\phi_2^-\} \\ \text{if } x_1^0<x_2^0}} + \underbrace{T\{\phi_1^-\phi_2^+\}}_{\substack{=N\{\phi_2^+\phi_1^-\} \\ \text{if } x_2^0<x_1^0}}$$

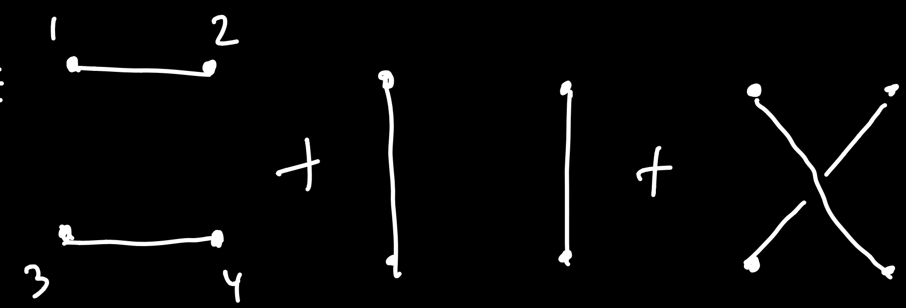
$$= \underbrace{\phi_1^+\phi_2^+ + \phi_1^-\phi_2^- + \phi_2^-\phi_1^+ + \phi_1^-\phi_2^+}_{=N\{\phi_1\phi_2\}} + \underbrace{\begin{cases}[\phi_1^+,\phi_2^-] & \text{for } x_1^0>x_2^0 \\ [\phi_2^+,\phi_1^-] & \text{for } x_2^0>x_1^0\end{cases}}_{\overbracket{\phi_1\phi_2}}$$

⌋

example 1

$$T\{\phi_1\phi_2\phi_3\phi_4\} = N\{\phi_1\phi_2\phi_3\phi_4 + \phi_1\phi_2\phi_3\phi_4 + \phi_1\phi_2\phi_3\phi_4 + \phi_1\phi_2\phi_3\phi_4$$
$$+ \phi_1\phi_2\phi_3\phi_4 + \phi_1\phi_2\phi_3\phi_4 + \phi_1\phi_2\phi_3\phi_4$$
$$+ \phi_1\phi_2\phi_3\phi_4 + \phi_1\phi_2\phi_3\phi_4 + \phi_1\phi_2\phi_3\phi_4\}$$

$$\Rightarrow \langle 0|T\{\phi_1\phi_2\phi_3\phi_4\}|0\rangle = D_F(x_1-x_2)\,D_F(x_3-x_4)$$
$$+ D_F(x_1-x_3)\,D_F(x_2-x_4)$$
$$+ D_F(x_1-x_4)\,D_F(x_2-x_3)$$

$\equiv$ 

"Feynman diagrams"

example 2 :

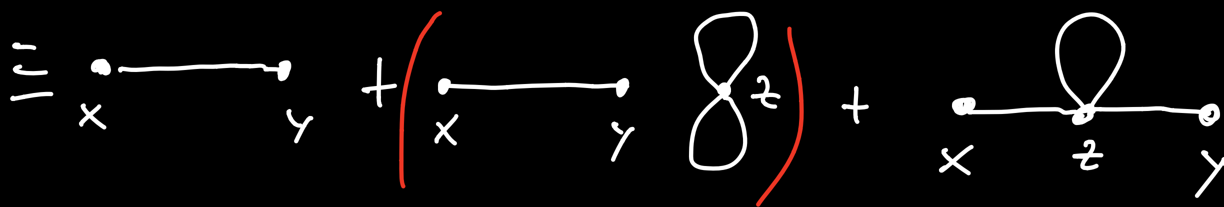
$$\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle \propto \langle 0|T\{\phi(x)\phi(y)\exp[-i\int_{-T}^{T} d^4z \frac{\lambda}{4!}\phi^4(z)]\}|0\rangle$$

$$= \underbrace{\langle 0|T\{\phi(x)\phi(y)\}|0\rangle}_{D_F(x-y)} - i\frac{\lambda}{4!}\int d^4z \langle 0|T\{\phi(x)\phi(y)\phi(z)^4\}|0\rangle$$

$$\phi(z)^4 = \phi_z\phi_z\phi_z\phi_z$$

$$= D_F(x-y) - i\frac{\lambda}{4!}\int d^4z\, D_F(x-y)\,D_F(z-z)^2 \times 3 \quad (3 \text{ possibilities to}$$

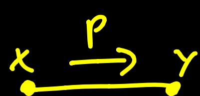
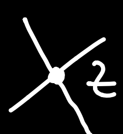
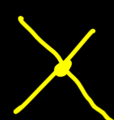
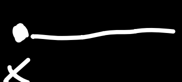
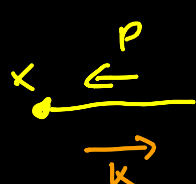
$$-i\frac{\lambda}{4!}\int d^4z\, D_F(x-z)\,D_F(y-z)\,D_F(z-z) \times 4 \times 3$$ (contract ϕ_z^4)



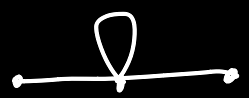
Feynman rules for ϕ^4 theory

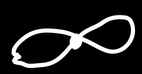
$$\boxed{\begin{aligned}&\langle 0|T\{\phi(x_1)\ldots\phi(x_n)\,\exp[-i\int dt\, H_I(t)]\}|0\rangle\\ &= \text{Sum of all possible diagrams with } n \text{ external points}\end{aligned}}$$

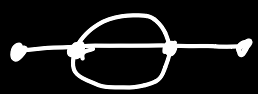
where (for ϕ^4 theory)

position space	momentum space
1. for each propagator	
 $= D_F(x-y)$	 $= \dfrac{i}{p^2-m^2+i\epsilon}$
2. for each "vertex" (internal points)	
 $= (-i\lambda)\int d^4z$	 $= -i\lambda$
3. for external point:	
 $= 1$	 $= e^{-ipx}$, e^{+ipx}

4. Divide by symmetry factor
$\equiv$ number of ways of inter-changing components without changing the diagrams

e.g.  $S = 2$ $(z \leftrightarrow z)$

 $S = 2^3 = 8$

 $S = 3! = 6$

$\vdots$

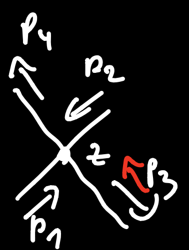
in case of doubt: count equivalent contractions!

5. Impose 4-momentum conservation @ each vertex

6. integrate over all (undetermined) momenta

Feynman rules in momentum space

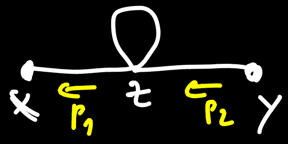
$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\varepsilon} e^{-ip\cdot(x-y)} = D_F(x-y)$$

vertex: 

$$= -i\lambda \int d^4z \, e^{-ip_1 z} e^{-ip_2 z} e^{\mp ip_3 z} e^{+ip_3 z}$$

$$= -i\lambda (2\pi)^4 \delta^{(4)}(p_1 + p_2 \pm p_3 - p_4)$$

example :



position space : $\frac{1}{2}(-i\lambda)\int d^4z\, D_F(x-z)\, D_F(z-y)\, D_F(z-z)$

$$= \frac{1}{2}(-i\lambda)\int d^4z \int \frac{d^4p_1}{(2\pi)^4}\int \frac{d^4p_2}{(2\pi)^4}\int \frac{d^4p_3}{(2\pi)^4} \times$$

$$\times \frac{i}{p_1^2-m^2+i\varepsilon}\;\frac{i}{p_2^2-m^2+i\varepsilon}\;\frac{i}{p_3^2-m^2+i\varepsilon} \times$$

$$\times\, e^{-ip_1(x-z)}\; e^{-ip_2(z-y)}\; \cancel{e^{-ip_3(z-z)}}$$

$$= \underbrace{\frac{1}{2}}_{4.}\underbrace{(-i\lambda)}_{2.}\underbrace{\int \frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_3}{(2\pi)^4}}_{6.}\;\underbrace{\frac{i}{p_1^2-m^2+i\varepsilon}\cdots\frac{i}{p_3^2-m^2+i\varepsilon}}_{1.} \times$$

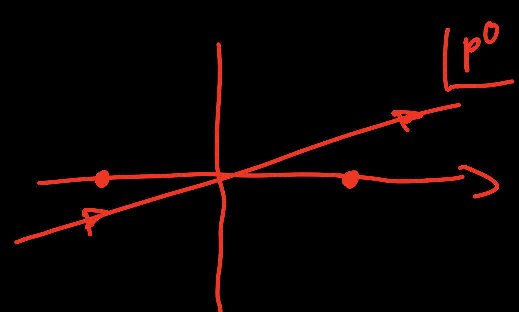
$$\times\, \underbrace{e^{-ip_1x}\, e^{+ip_2y}}_{3.}\;\underbrace{(2\pi)^4\,\delta^{(4)}(p_1-p_2)}_{5.}$$

NB : • $\int d^4z = \lim_{T\to\infty(1-i\varepsilon)} \int_{-T}^{T} dz^0 \int d^3z$

• $e^{ip\cdot z} \Rightarrow p\cdot z \supset p^0z^0$ must be real

$\Rightarrow p^0 \propto (1+i\varepsilon)$

$\Rightarrow$

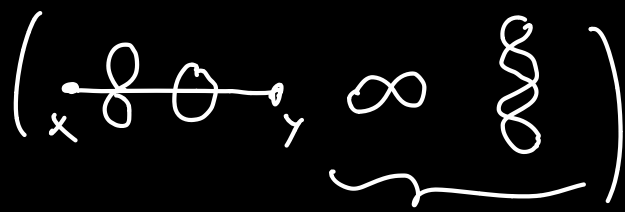


i.e same pole prescription as for Feynman propagator! ✓

Exponentiation of disconnected diagrams

typical diagram:

$$\langle 0| T\{\phi(x)\phi(y)\exp[-i\int dt\, H_I(t)]\}|0\rangle \supset$$



"disconnected pieces"

= no connection to external points (x or y)

label all disconnected pieces:

$$V_i \in \{\infty, 8, \dots\}$$

$\Rightarrow$ every diagram = (value of connected piece)

$$\times \prod_i \frac{1}{n_i!}(V_i)^{n_i}$$

symmetry factor = number of possibilies of arranging n_i identical pieces

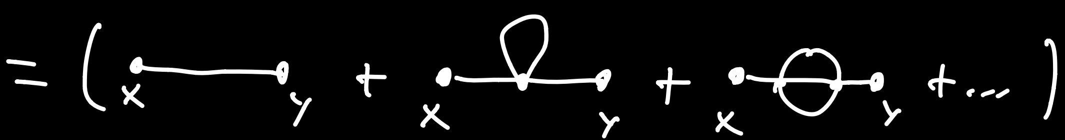
$$\Rightarrow \sum \text{all diagrams} = \sum_{\substack{\text{all possible}\\ \text{connected}\\ \text{pieces}}} \; \sum_{\{n_1,n_2,n_3,\dots\}} \begin{pmatrix}\text{value of}\\ \text{conn. piece}\end{pmatrix} \prod_i \frac{1}{n_i!}(V_i)^{n_i}$$

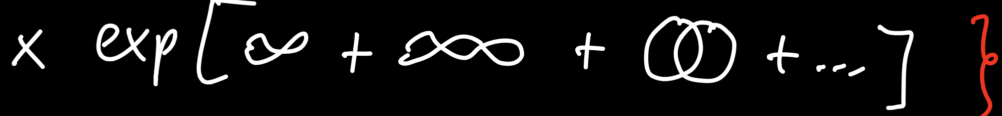
$$= \left(\sum \text{connected}\right) \times \underbrace{\sum_{\{n_1,n_2,\dots\}} \prod_i \frac{1}{n_i!}(V_i)^{n_i}}_{\prod_i \sum_{n_i=1}^{\infty} \frac{1}{n_i!}(V_i)^{n_i}}$$

$$= \prod_i \exp V_i = \exp \sum_i V^i$$

e.g. 2-point function:

$$\langle 0 | T\{\phi(x)\phi(y) \exp[-i\int dt\, H_I(t)]\} | 0 \rangle$$





"energy density of vacuum"

$$\Downarrow$$

$$\boxed{\langle \Omega | T\{\phi(x_1)\dots\phi(x_n)\} | \Omega \rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0 | T\{\phi_I(x_1)\dots\phi_I(x_n) \exp[-i\int_{-T}^{T} dt\, H_I(t)]\} | 0 \rangle}{\langle 0 | T\{\exp[-i\int_{-T}^{T} dt\, H_I(t)]\} | 0 \rangle} = \begin{pmatrix}\text{sum of all } \underline{\text{connected}} \text{ diagrams}\\ \text{with } n \text{ external points}\end{pmatrix}}$$