7. Cross sections and decay rates

cross section σ ~ effective target area seen by an interacting particle

$\propto$ probability for interaction to happen

introduce in 3 steps...

i)

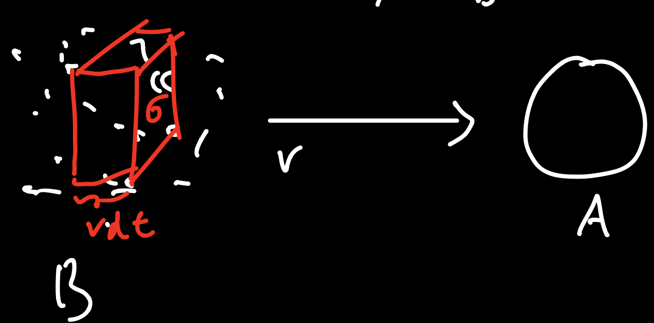
"hard sphere approximation":

scattering takes place if $b < r$

no scattering if $b > r$

$\Rightarrow \sigma = \pi b^2$

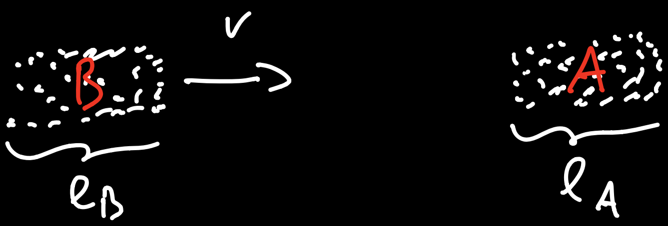
ii) many incident particles (number density n_B)



$\Rightarrow$ # events $= n_B \cdot \sigma \, v dt$

$$\leadsto \boxed{\sigma \equiv \frac{\#\text{events}/\text{time}}{v \cdot n_B}}$$

iii) two colliding beams:



$\Rightarrow$ # events $\overset{!}{\propto}$ $\ell_B \ell_A n_B n_A A$ ↖ overlapping area of two beams

$$\rightsquigarrow \boxed{\sigma \equiv \frac{\#\text{ events}}{n_A n_B \ell_A \ell_B A} = A \frac{\#\text{ events}}{N_A N_B}}$$

- $N_A = 1$
- $N_B = n_B \cdot A \cdot \underbrace{\ell_B}_{v \cdot dt}$

$\rightsquigarrow$ case ii) ✓

<u>decay rate</u>

$$\Gamma \equiv \frac{\#(\text{decays/time})}{\#\text{ particles still left}}$$

$$= -\frac{dn/dt}{n} \Rightarrow n = n_0 e^{-\Gamma t}$$

$\rightsquigarrow$ <u>lifetime</u> $\tau \equiv \frac{1}{\Gamma}$

The S-matrix

general wave packet : $|\phi\rangle = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} \phi(\vec{k}) \, |\vec{k}\rangle$

one-particle states of momentum $\vec{k}$ in interacting (!) theory

$\rightsquigarrow$ want to compute "scattering probability" for $A + B \longrightarrow \ldots$:

$$P = |\underbrace{\langle \phi_1 \phi_2 \ldots}_{\text{"out-state"}} | \underbrace{\phi_A \phi_B \rangle}_{\text{"in-state"}}|^2$$

set up in distant future

set up in remote past

BUT: still in Heisenberg picture!
[states are t-independent, but operators – and hence their eigenvalues, like $\vec{k}$ – are time-dependent!]

in-state : will consider highly concentrated wave packages "particles"

out-state : " " plane waves
~ what is measured by detectors

Definition of "S-matrix"

$$\boxed{{}_{out}\langle \vec{p}_1 \vec{p}_2 \cdots | \vec{k}_A \vec{k}_B \rangle_{in} \equiv \langle \vec{p}_1 \vec{p}_2 \cdots | S | \vec{k}_A \vec{k}_B \rangle}$$

constructed @ any common reference time

expected structure:

$$S = 1 + iT$$

no scattering

"T-matrix" $\propto \delta^{(4)}(k_A + k_B - \sum_f p_f)$

$\Rightarrow$ Def. "invariant matrix element $\mathcal{M}$": [~ scattering amplitude in QM]

$$\boxed{\langle \vec{p}_1 \vec{p}_2 \cdots | iT | \vec{k}_A \vec{k}_B \rangle \equiv (2\pi)^4 \delta^{(4)}(k_A + k_B - \sum_f p_f) \cdot i\mathcal{M}(k_A k_B \to p_f)}$$

from $\mathcal{M}$ to σ, Γ

consider [single] target A, many incident particles B:

$\Rightarrow$ initial state:

$$|\phi_A \phi_B\rangle_{in} = \int \frac{d^3k_A}{(2\pi)^3} \int \frac{d^3k_B}{(2\pi)^3} \frac{\phi_A(\vec{k}_A)\, \phi_B(\vec{k}_B)}{2\sqrt{E_A E_B}}\, e^{i\vec{k}_B\cdot\vec{b}}\, |k_A k_B\rangle$$

number of scattering events:

$$N = \underbrace{\int d^2b \frac{N_B}{A_{rea}}}_{\text{\# particles within range of b}} P(AB \to 1, 2, \dots, n)$$

assume constant over range of interaction

$$= \frac{N_B}{A_{rea}} \int d^2b\, \underbrace{\prod_{f=1}^{n} \int \frac{d^3p_f}{(2\pi)^3} \frac{1}{2E_f}}_{\text{"sum over all possible momentum configurations in final state"}} \left|{}_{out}\langle p_1 p_2 \cdots p_n | \phi_A \phi_B\rangle\right|^2$$

$$\Rightarrow d\sigma = \frac{A_{rea}}{N_B} \frac{dN}{(N_A = 1)} = \left(\prod_f \frac{d^3p}{(2\pi)^3} \frac{1}{2E_f}\right) \int d^2b \prod_{i=A,B} \int \frac{d^3k_i}{(2\pi)^3} \frac{\phi_i(\vec{k}_i)}{\sqrt{2E_i}}$$

w.r.t. final configuration

$$\int \frac{d^3k_i'}{(2\pi)^3} \frac{\phi_i(\vec{k}_i')}{\sqrt{2E_i'}}\, e^{i\vec{b}\cdot(\vec{k}_B' - \vec{k}_B)} \times$$

$$\int d^2b \to (2\pi)^2 \delta^{(2)}(\vec{k}_B^{\perp} - \vec{k}_B'^{\perp})$$

$$\times \; {}_{out}\langle \vec{p}_1 \cdots \vec{p}_n | \vec{k}_A \vec{k}_B \rangle_{in} \left({}_{out}\langle \vec{p}_1 \cdots \vec{p}_n | \vec{k}_A' \vec{k}_B' \rangle_{in} \right)^*$$

$$\downarrow$$

$$i\mathcal{M}(\vec{k}_A \vec{k}_B \to \{\vec{p}_f\})(2\pi)^4 \delta^{(4)}(k_A + k_B - \sum_f p_f)$$

- 12 integrals, 10 δ-functions
- recall that ϕ_i are highly localized in $\vec{k}$-space ⇝ pull outside integrals
- ...

$$\boxed{d\sigma = \frac{1}{2E_A \, 2E_B |v_A - v_B|} \, d\Pi_n \; |\mathcal{M}(p_A p_B \to \{p_f\})|^2}$$

"relativistically invariant n-body phase space"

$$\boxed{d\Pi_n = \left(\prod_f \frac{d^3p}{(2\pi)^3} \frac{1}{2E_f} \right) (2\pi)^4 \delta^{(4)}(k_A + k_B - \sum p_f)}$$

↳ purely kinematical object, can be computed "once and for all"

[$\mathcal{M}$ encodes the dynamics, i.e. the model-dependent part of the interactions]

prefactor : $(E_A E_B |v_A - v_B|)^{-1} = |E_B k_A - E_A k_B|^{-1} = |\varepsilon_{\mu x y \nu} K_A^\mu K_B^\nu|^{-1}$

(...) ⇝ same transformation properties as an area in z-direction!

(e.g. invariant und boost along z-direction)

similar: decay rate $\boxed{d\Gamma = \frac{1}{2m_A} d\Pi_n |\mathcal{M}(m_A \to \{p_f\})|^2}$

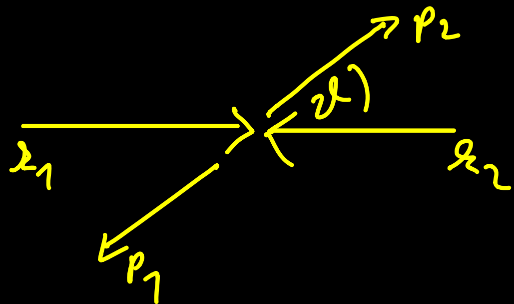
How to treat identical final-state particles?

a) restrict $\int d\Pi_n$ to physically inequivalent configurations

b) integrate over all sets of $\{\vec{p}_f\}$ and then divide by $(n!)$.

example: 2-body final state in center-of-mass system [CMS]

- $\sum \vec{k}_i + \sum \vec{p}_f = 0$
- $\sum k_i^0 = \sum p_f^0 \stackrel{!}{=} E_{cm}$



$$\Rightarrow \int d\Pi_2 = \int \frac{d^3p_1}{(2\pi)^3}\frac{d^3p_2}{(2\pi)^3}\frac{1}{2E_1}\frac{1}{2E_2}(2\pi)^4\delta^{(4)}(k_1+k_2-p_1-p_2)$$

$$= \int \frac{\overbrace{d^3p_1}^{=p_1^2\,d\Omega\,dp_1}}{(2\pi)^3}\frac{1}{2E_1}\frac{1}{2E_2}(2\pi)\delta(E_{cm}-E_1-E_2)$$

$E_2 = E_2(\vec{k}_1,\vec{k}_2,\vec{p}_1)$

- $E_i = \sqrt{|\vec{p}_i|^2 + m_i^2}$
- $\delta(f(x)) = \frac{1}{|f'(x_0)|}\delta(x-x_0)$

$$= \int \frac{d\Omega}{(2\pi)^2} \frac{|\vec{p}_1|^2}{4E_1E_2} \left| \underbrace{-\frac{dE_1}{dp_1}}_{\overset{\shortparallel}{-\frac{|\vec{p}_1|}{E_1}}} - \underbrace{\frac{dE_2}{dp_1}}_{-\frac{|\vec{p}_1|}{E_2}} \right|^{-1}$$

$$= \int \frac{d\Omega}{(2\pi)^2} \frac{|\vec{p}_1|}{4} |\underbrace{E_1 + E_2}_{E_{cm}}|^{-1}$$

$$\Rightarrow \boxed{\int d\Pi_2 = \int d\Omega \, \frac{1}{16\pi^2} \frac{|\vec{p}_1|}{E_{cm}}} \quad \text{in CMS frame}$$

$$\Rightarrow \left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{1}{4E_AE_B|v_A - v_B|} \frac{|\vec{p}_1|}{(2\pi)^2 4E_{cm}} |M|^2$$

if $|M|^2$ is symmetric about collision axis :

$$\int d\Pi_2 = \int d\cos\vartheta \, \frac{1}{8\pi} \frac{\vec{p}_1}{E_{cm}}$$

$$\int = \begin{cases} \int_{-1}^{+1} & \text{for distinguishable final-state particles} \\ \int_{0}^{+1} & \text{for identical final-state particles} \end{cases}$$